Calculus 2

NINTH EDITION

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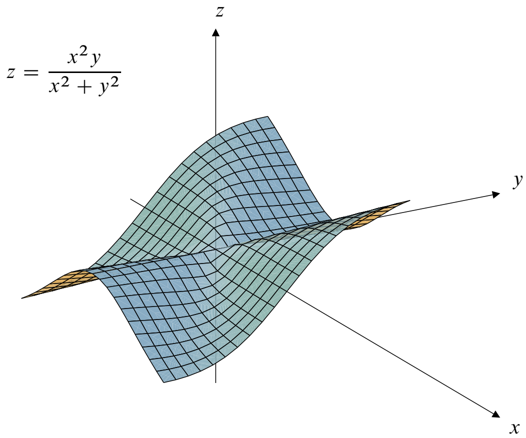

Figure 12.15 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0$

DEFINITION 3

The function $f(x, y)$ is continuous at the point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

It remains true that sums, differences, products, quotients, and compositions of continuous functions are continuous. The functions of Examples 3 and 4 above are continuous wherever they are defined, that is, at all points except the origin. There is no way to define $f(0, 0)$ so that these functions become continuous at the origin. They show that the continuity of the single-variable functions $f(x, b)$ at $x = a$ and $f(a, y)$ at $y = b$ does *not* imply that $f(x, y)$ is continuous at (a, b). In fact, even if $f(x, y)$ is continuous along every straight line through (a, b), it still need not be continuous at (a, b). (See Exercises 16–17 below.) Note, however, that the function $f(x, y)$ of Example 5, although not defined at the origin, has a continuous extension to that point. If we extend the domain of f by defining $f(0, 0) = \lim_{(x,y)\to(0,0)} f(x, y) = 0$, then f is continuous on the whole xy-plane.

As for functions of one variable, the existence of a limit of a function at a point does not imply that the function is continuous at that point. The function

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

satisfies $\lim_{(x,y)\to(0,0)} f(x, y) = 0$, which is not equal to $f(0, 0)$, so f is not continuous at $(0, 0)$. Of course, we can *make* f continuous at $(0, 0)$ by redefining its value at that point to be 0.

EXERCISES 12.2

In Exercises 1–12, evaluate the indicated limit or explain why it does not exist.

1. $\displaystyle\lim_{(x,y)\to(2,-1)} xy + x^2$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2}$

3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{y}$

4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2 + y^2}$

5. $\displaystyle\lim_{(x,y)\to(1,\pi)} \frac{\cos(xy)}{1 - x - \cos y}$

6. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2(y - 1)^2}{x^2 + (y - 1)^2}$

7. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2 + y^2}$

8. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x - y)}{\cos(x + y)}$

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2 + y^2}$

10. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{2x^2 - xy}{4x^2 - y^2}$

11. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^4}$

12. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{2x^4 + y^4}$

13. How can the function

$$f(x, y) = \frac{x^2 + y^2 - x^3 y^3}{x^2 + y^2}, \qquad (x, y) \neq (0, 0),$$

be defined at the origin so that it becomes continuous at all points of the xy-plane?

14. How can the function

$$f(x, y) = \frac{x^3 - y^3}{x - y}, \qquad (x \neq y),$$

be defined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane?

15. What is the domain of

$$f(x, y) = \frac{x - y}{x^2 - y^2}?$$

Does $f(x, y)$ have a limit as $(x, y) \to (1, 1)$? Can the domain of f be extended so that the resulting function is continuous at $(1, 1)$? Can the domain be extended so that the resulting function is continuous everywhere in the xy-plane?

16. Given a function $f(x, y)$ and a point (a, b) in its domain, define single-variable functions g and h as follows:

$$g(x) = f(x, b), \qquad h(y) = f(a, y).$$

If g is continuous at $x = a$ and h is continuous at $y = b$, does it follow that f is continuous at (a, b)? Conversely, does the continuity of f at (a, b) guarantee the continuity of g at a and the continuity of h at b? Justify your answers.

17. Let $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ be a unit vector, and let

$$f_{\mathbf{u}}(t) = f(a + tu, b + tv)$$

be the single-variable function obtained by restricting the domain of $f(x, y)$ to points of the straight line through (a, b) parallel to $\mathbf{u}$. If $f_{\mathbf{u}}(t)$ is continuous at $t = 0$ for every unit vector $\mathbf{u}$, does it follow that f is continuous at (a, b)? Conversely, does the continuity of f at (a, b) guarantee the continuity of $f_{\mathbf{u}}(t)$ at $t = 0$? Justify your answers.

18. What condition must the nonnegative integers m, n, and p satisfy to guarantee that $\lim_{(x,y)\to(0,0)} x^m y^n/(x^2 + y^2)^p$ exists? Prove your answer.

19. What condition must the constants a, b, and c satisfy to guarantee that $\lim_{(x,y)\to(0,0)} xy/(ax^2 + bxy + cy^2)$ exists? Prove your answer.

20. Can the function $f(x, y) = \dfrac{\sin x \sin^3 y}{1 - \cos(x^2 + y^2)}$ be defined at $(0, 0)$ in such a way that it becomes continuous there? If so, how?

21. Use two- and three-dimensional mathematical graphing software to examine the graph and level curves of the function $f(x, y)$ of Example 3 on the region $-1 \le x \le 1$, $-1 \le y \le 1$, $(x, y) \ne (0, 0)$. How would you describe the behaviour of the graph near $(x, y) = (0, 0)$?

22. Use two- and three-dimensional mathematical graphing software to examine the graph and level curves of the function $f(x, y)$ of Example 4 on the region $-1 \le x \le 1$, $-1 \le y \le 1$, $(x, y) \ne (0, 0)$. How would you describe the behaviour of the graph near $(x, y) = (0, 0)$?

23. The graph of a single-variable function $f(x)$ that is continuous on an interval is a curve that has no *breaks* in it there and that intersects any vertical line through a point in the interval exactly once. What analogous statement can you make about the graph of a bivariate function $f(x, y)$ that is continuous on a region of the xy-plane?

24. (a) State explicitly the version of Definition 2 that applies to a function f of a single variable x.

(b) Let f be a function with domain the set of numbers $1/n$ for $n = 1, 2, 3, \ldots$ and having values given by $f(1/n) = (n - 1)/n$. According to part (a) does $\lim_{x\to 1} f(x)$ exist? What about $\lim_{x\to 0} f(x)$? Evaluate whichever of these limits does exist.

(c) Which of the two limits in (*b*) exist by Definition 8 in Section 1.5?

12.3 Partial Derivatives

In this section we begin the process of extending the concepts and techniques of single-variable calculus to functions of more than one variable. It is convenient to begin by considering the rate of change of such functions with respect to one variable at a time. Thus, a function of n variables has n *first-order partial derivatives,* one with respect to each of its independent variables. For a function of two variables, we make this precise in the following definition:

DEFINITION 4

The **first partial derivatives** of the function $f(x, y)$ **with respect to the variables** x and y are the functions $f_1(x, y)$ and $f_2(x, y)$ given by

$$f_1(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

$$f_2(x, y) = \lim_{k\to 0} \frac{f(x, y + k) - f(x, y)}{k},$$

provided these limits exist.

Each of the two partial derivatives is the limit of a Newton quotient in one of the variables. Observe that $f_1(x, y)$ is just the ordinary first derivative of $f(x, y)$ considered as a function of x only, regarding y as a constant parameter. Similarly, $f_2(x, y)$ is the first derivative of $f(x, y)$ considered as a function of y alone, with x held fixed.

EXAMPLE 1 If $f(x, y) = x^2 \sin y$, then

$$f_1(x, y) = 2x \sin y \qquad \text{and} \qquad f_2(x, y) = x^2 \cos y.$$

The subscripts 1 and 2 in the notations for the partial derivatives refer to the first and second variables of f. For functions of one variable we use the notation f' for the derivative; the *prime* ($'$) denotes differentiation with respect to the only variable on which f depends. For functions f of two variables, we use f_1 or f_2 to show the variable of differentiation. Do not confuse these subscripts with subscripts used for other purposes (e.g., to denote the components of vectors).

The partial derivative $f_1(a, b)$ measures the rate of change of $f(x, y)$ with respect to x at $x = a$ while y is held fixed at b. In graphical terms, the surface $z = f(x, y)$ intersects the vertical plane $y = b$ in a curve. If we take horizontal and vertical lines through the point $(0, b, 0)$ as coordinate axes in the plane $y = b$, then the curve has equation $z = f(x, b)$, and its slope at $x = a$ is $f_1(a, b)$. (See Figure 12.16.) Similarly, $f_2(a, b)$ represents the rate of change of f with respect to y at $y = b$ with x held fixed at a. The surface $z = f(x, y)$ intersects the vertical plane $x = a$ in a curve $z = f(a, y)$ whose slope at $y = b$ is $f_2(a, b)$. (See Figure 12.17.)

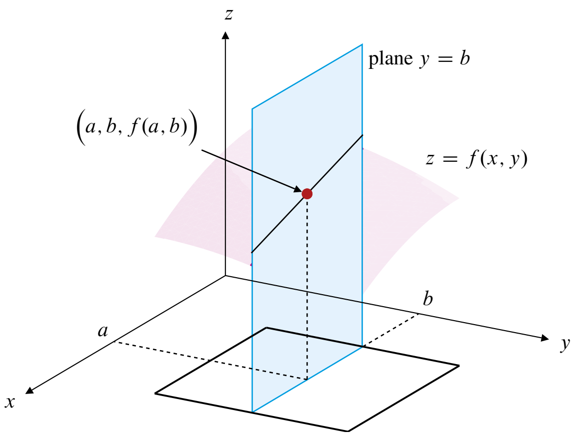

Figure 12.16 $f_1(a, b)$ is the slope of the red curve of intersection of the red surface $z = f(x, y)$ and the blue vertical plane $y = b$ at $x = a$

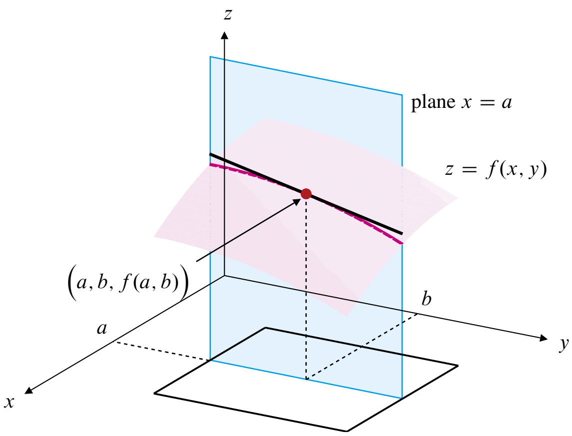

Figure 12.17 $f_2(a, b)$ is the slope of the red curve of intersection of the red surface $z = f(x, y)$ and the blue vertical plane $x = a$ at $y = b$

Various notations can be used to denote the partial derivatives of $z = f(x, y)$ considered as functions of x and y:

Notations for first partial derivatives

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_1(x, y) = D_1 f(x, y)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_2(x, y) = D_2 f(x, y)$$

The symbol $\partial/\partial x$ should be read as "partial with respect to x" so $\partial z/\partial x$ is "partial z with respect to x." The reason for distinguishing ∂ (pronounced "die") from the d of ordinary derivatives of single-variable functions will be made clear later. Similar notations can be used to denote the values of partial derivatives at a particular point (a, b):

Values of partial derivatives

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \left.\left(\frac{\partial}{\partial x} f(x,y)\right)\right|_{(a,b)} = f_1(a,b) = D_1 f(a,b)$$

$$\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \left.\left(\frac{\partial}{\partial y} f(x,y)\right)\right|_{(a,b)} = f_2(a,b) = D_2 f(a,b)$$

BEWARE! Read the paragraph at the right carefully. It explains why, at least for the time being, we are using subscripts 1 and 2 instead of subscripts x and y for the partial derivatives of $f(x, y)$. Later on, and especially when we are discussing partial differential equations or dealing with vector-valued functions for which numerical subscripts normally represent components, we will prefer to use letter subscripts for partial derivatives.

Some authors prefer to use f_x, $D_x f$, or $\partial f/\partial x$, and f_y, $D_y f$, or $\partial f/\partial y$, instead of f_1 and f_2. However, this can lead to problems of ambiguity when compositions of functions arise. For instance, suppose $f(x, y) = x^2 y$. What should $f_x(x^2, xy)$ mean? By $f_1(x^2, xy)$ we clearly mean to evaluate the partial derivative of $f(u, v) = u^2 v$ with respect to its first variable u and evaluate the result at $u = x^2$ and $v = xy$:

$$f_1(x^2, xy) = \left.\left(\frac{\partial}{\partial u} f(u,v)\right)\right|_{u=x^2, v=xy} = 2uv\Big|_{u=x^2, v=xy} = (2)(x^2)(xy) = 2x^3 y.$$

But does $f_x(x^2, xy)$ mean the same thing? One could argue that

$$f_x(x^2, xy) = \frac{\partial}{\partial x}\Big(f(x^2, xy)\Big) = \frac{\partial}{\partial x}\Big((x^2)^2(xy)\Big) = \frac{\partial}{\partial x}(x^5 y) = 5x^4 y.$$

In order to avoid such ambiguities we usually prefer to use f_1 and f_2 instead of f_x and f_y. (However, in some situations where no confusion is likely to occur we may still use the notations f_x and f_y, and also $D_x f$, $D_y f$, $\partial f/\partial x$, and $\partial f/\partial y$.)

All the standard differentiation rules for sums, products, reciprocals, and quotients continue to apply to partial derivatives.

EXAMPLE 2 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^3 y^2 + x^4 y + y^4$.

Solution $\partial z/\partial x = 3x^2 y^2 + 4x^3 y$ and $\partial z/\partial y = 2x^3 y + x^4 + 4y^3$.

EXAMPLE 3 Find $f_1(0, \pi)$ if $f(x, y) = e^{xy}\cos(x + y)$.

Solution

$$f_1(x, y) = y\, e^{xy}\cos(x + y) - e^{xy}\sin(x + y),$$
$$f_1(0, \pi) = \pi\, e^0\cos(\pi) - e^0\sin(\pi) = -\pi.$$

The single-variable version of the Chain Rule also continues to apply to, say, $f\big(g(x, y)\big)$, where f is a function of only one variable having derivative f':

$$\frac{\partial}{\partial x} f\big(g(x,y)\big) = f'\big(g(x,y)\big)\, g_1(x,y), \qquad \frac{\partial}{\partial y} f\big(g(x,y)\big) = f'\big(g(x,y)\big)\, g_2(x,y).$$

We will develop versions of the Chain Rule for more complicated compositions of multivariate functions in Section 12.5.

EXAMPLE 4 If f is an everywhere differentiable function of one variable, show that $z = f(x/y)$ satisfies the *partial differential equation*

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

Solution By the (single-variable) Chain Rule,

$$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right) \quad \text{and} \quad \frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right).$$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(x \times \frac{1}{y} + y \times \frac{-x}{y^2}\right) = 0.$$

Definition 4 can be extended in the obvious way to cover functions of more than two variables. If f is a function of n variables $x_1, x_2, \dots, x_n$, then f has n first partial derivatives, $f_1(x_1, x_2, \dots, x_n)$, $f_2(x_1, x_2, \dots, x_n)$, ..., $f_n(x_1, x_2, \dots, x_n)$, one with respect to each variable.

EXAMPLE 5

$$\frac{\partial}{\partial z}\left(\frac{2xy}{1 + xz + yz}\right) = -\frac{2xy}{(1 + xz + yz)^2}(x + y).$$

Again, all the standard differentiation rules are applied to calculate partial derivatives.

Remark If a single-variable function $f(x)$ has a derivative $f'(a)$ at $x = a$, then f is necessarily continuous at $x = a$. This property does *not* extend to partial derivatives. Even if all the first partial derivatives of a function of several variables exist at a point, the function may still fail to be continuous at that point. See Exercise 36 below.

Tangent Planes and Normal Lines

If the graph $z = f(x, y)$ is a "smooth" surface near the point P with coordinates $\big(a, b, f(a, b)\big)$, then that graph will have a **tangent plane** and a **normal line** at P. The normal line is the line through P that is perpendicular to the surface; for instance, a line joining a point on a sphere to the centre of the sphere is normal to the sphere. Any nonzero vector that is parallel to the normal line at P is called a normal vector to the surface at P. The tangent plane to the surface $z = f(x, y)$ at P is the plane through P that is perpendicular to the normal line at P.

Let us assume that the surface $z = f(x, y)$ has a *nonvertical* tangent plane (and therefore a *nonhorizontal* normal line) at point P. (Later in this chapter we will state precise conditions that guarantee that the graph of a function has a nonvertical tangent plane at a point.) The tangent plane intersects the vertical plane $y = b$ in a straight line that is tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $y = b$. (See Figures 12.16 and 12.18.) This line has slope $f_1(a, b)$, so it is parallel to the vector $\mathbf{T}_1 = \mathbf{i} + f_1(a, b)\mathbf{k}$. Similarly, the tangent plane intersects the vertical plane $x = a$ in a straight line having slope $f_2(a, b)$. This line is therefore parallel to the vector $\mathbf{T}_2 = \mathbf{j} + f_2(a, b)\mathbf{k}$. It follows that the tangent plane, and therefore the surface $z = f(x, y)$ itself, has normal vector

$$\mathbf{n} = \mathbf{T}_2 \times \mathbf{T}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & f_2(a, b) \\ 1 & 0 & f_1(a, b) \end{vmatrix} = f_1(a, b)\mathbf{i} + f_2(a, b)\mathbf{j} - \mathbf{k}.$$

A normal vector to $z = f(x, y)$ **at** $\big(a, b, f(a, b)\big)$ **is**

$\mathbf{n} = f_1(a, b)\mathbf{i} + f_2(a, b)\mathbf{j} - \mathbf{k}.$

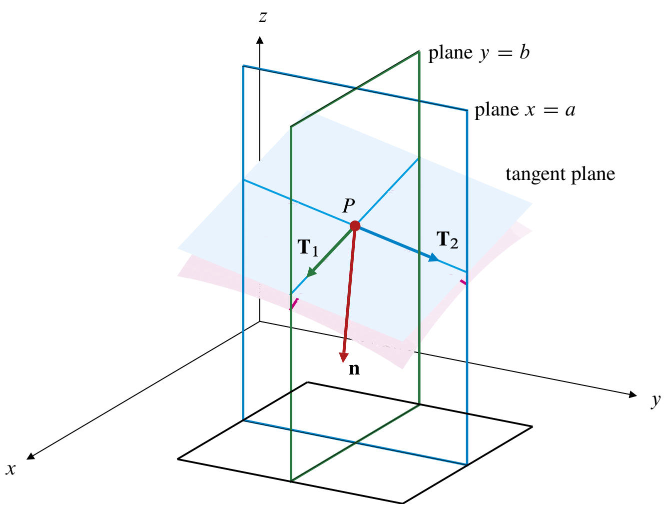

Figure 12.18 The tangent plane and a normal vector to $z = f(x, y)$ at $P = \big(a, b, f(a, b)\big)$. In this figure the graph of f is red, the tangent plane is blue, and the normal to both at P is red. The normal is the cross product of the tangent vectors (T_2) in the blue vertical plane $x = a$ and (T_1) in the green vertical plane $y = b$.

Since the tangent plane passes through $P = (a, b, f(a, b))$, it has equation

$$f_1(a, b)(x - a) + f_2(a, b)(y - b) - (z - f(a, b)) = 0,$$

or, equivalently,

An equation of the tangent plane to $z = f(x, y)$ at $\big(a, b, f(a, b)\big)$ is

$$z = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b).$$

We shall obtain this result by a different method in Section 12.7.

The normal line to $z = f(x, y)$ at $\big(a, b, f(a, b)\big)$ has direction vector $f_1(a, b)\mathbf{i} + f_2(a, b)\mathbf{j} - \mathbf{k}$ and so has equations

$$\frac{x - a}{f_1(a, b)} = \frac{y - b}{f_2(a, b)} = \frac{z - f(a, b)}{-1}$$

with suitable modifications if either $f_1(a, b) = 0$ or $f_2(a, b) = 0$.

EXAMPLE 6 Find a normal vector and equations of the tangent plane and normal line to the graph $z = \sin(xy)$ at the point where $x = \pi/3$ and $y = -1$.

Solution The point on the graph has coordinates $(\pi/3, -1, -\sqrt{3}/2)$. Now

$$\frac{\partial z}{\partial x} = y\cos(xy) \qquad \text{and} \qquad \frac{\partial z}{\partial y} = x\cos(xy).$$

At $(\pi/3, -1)$ we have $\partial z/\partial x = -1/2$ and $\partial z/\partial y = \pi/6$. Therefore, the surface has normal vector $\mathbf{n} = -(1/2)\mathbf{i} + (\pi/6)\mathbf{j} - \mathbf{k}$ and tangent plane

$$z = \frac{-\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\pi}{6}(y + 1),$$

or, more simply, $3x - \pi y + 6z = 2\pi - 3\sqrt{3}$. The normal line has equation

$$\frac{x - \frac{\pi}{3}}{\frac{-1}{2}} = \frac{y + 1}{\frac{\pi}{6}} = \frac{z + \frac{\sqrt{3}}{2}}{-1} \quad \text{or} \quad \frac{6x - 2\pi}{-3} = \frac{6y + 6}{\pi} = \frac{6z + 3\sqrt{3}}{-6}.$$