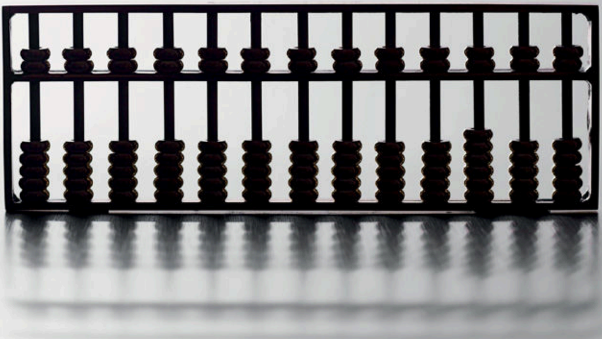
Arab World Edition

Introductory Mathematical Analysis

for Business, Economics, and the Life and Social Sciences



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ALWAYS LEARNING PEARSON

provided that $|(1+r)^{-1}| < 1$. If the rate r is positive, then $1 < 1+r$ so that $0 < (1+r)^{-1} = \frac{1}{1+r} < 1$ and the proviso is satisfied.[7]

In practical terms, this means that if an amount R/r is invested at time 0 in an account that bears interest at the rate of r per payment period, then R can be withdrawn at times $1, 2, \ldots, k, \ldots$ indefinitely. It is easy to see that this makes sense because if R/r is invested at time 0, then at time 1 it is worth $(R/r)(1+r) = R/r + R$. If, at time 1, R is withdrawn, then $R/r + R - R = R/r$ remains and this process can be continued indefinitely so that at any time k, the amount after the kth withdrawal is still R/r. In other words, the withdrawals R are such that they consume only the interest earned since the last withdrawal and leave the principal intact. Well-managed endowment funds are run this way. The amount withdrawn each year to fund a scholarship, say, should not exceed the amount earned in interest during the previous year.

EXAMPLE 51 Present Value of a Perpetuity

Middle East University would like to establish a scholarship worth \$15,000 to be awarded to the first year Business student who attains the highest grade in Math 101, Business Mathematics. The award is to be made annually, and the Vice President Finance believes that, for the foreseeable future, the university will be able to earn at least 2% a year on investments. What principal is needed to ensure the viability of the scholarship?

Solution: The university needs to fund a perpetuity with payments $R = 15{,}000$ and annual interest rate $r = 0.02$. It follows that $\$15{,}000/0.02 = \$750{,}000$ is needed.

Now work Problem 5 ◁

Limits

An infinite sum, such as $\sum_{k=1}^{\infty} R(1+r)^{-k}$, which has arisen here, derives its meaning from the associated *finite* partial sums. Here the nth partial sum is $\sum_{k=1}^{n} R(1+r)^{-k}$, which we recognize as $Ra_{\overline{n}|r}$, the present value of the annuity consisting of n equal payments of R at an interest rate of r per payment period.

Let $(c_k)_{k=1}^{\infty}$ be an infinite sequence as in Section 4.1. We say that L is the *limit of a sequence* and write

$$\lim_{k\to\infty} c_k = L$$

if *we can make the values c_k as close as we like to L by taking k sufficiently large*. The equation can be read as "the limit of c_k as k goes to infinity is equal to L." A sequence can fail to have a limit but it can have at most one limit so that we speak of "the limit."

We have already met an important example of this concept. In Section 3.1 we defined the number e as the smallest real number which is greater than all of the real numbers $e_n = \left(\frac{n+1}{n}\right)^n$, for n any positive integer. In fact, we have also

$$\lim_{n\to\infty} e_n = e$$

A general infinite sequence $(c_k)_{k=1}^{\infty}$ determines a new sequence $(s_n)_{n=1}^{\infty}$, where $s_n = \sum_{k=1}^{n} c_k$. We define

$$\sum_{k=1}^{\infty} c_k = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} c_k$$

[7] The proviso is also satisfied for $r < -2$.

This agrees with what we said about the sum of an infinite geometric sequence in Section 4.1, and it is important to realize that the sums which arise for the present values of annuities and perpetuities are but special cases of sums of geometric sequences.

However, we wish to make a simple observation by combining some of the equalities of this section:

$$\frac{R}{r} = \sum_{k=1}^{\infty} R(1+r)^{-k} = \lim_{n\to\infty} \sum_{k=1}^{n} R(1+r)^{-k} = \lim_{n\to\infty} Ra_{\overline{n}|r}$$

and, taking $R = 1$, we get

$$\lim_{n\to\infty} a_{\overline{n}|r} = \frac{1}{r}$$

We can verify this observation directly. In the defining equation

$$a_{\overline{n}|r} = \frac{1-(1+r)^{-n}}{r}$$

only $(1+r)^{-n} = 1/(1+r)^n$ depends on n. Because $1+r > 1$, we can make the values $(1+r)^n$ as large as we like by taking n sufficiently large. It follows that we can make the values $1/(1+r)^n$ as close as we like to 0 by taking n sufficiently large. It follows that in the definition of $a_{\overline{n}|r}$, we can make the numerator as close as we like to 1 by taking n sufficiently large and hence that we can make the whole fraction as close as we like to $1/r$ by taking n sufficiently large.

EXAMPLE 52 Limit of a Sequence

Find $\lim_{n\to\infty} \frac{2n^2+1}{3n^2-5}$.

Solution: First we rewrite the fraction $\frac{2n^2+1}{3n^2-5}$.

$$\lim_{n\to\infty} \frac{2n^2+1}{3n^2-5} = \lim_{n\to\infty} \frac{n^2\left(2+\frac{1}{n^2}\right)}{n^2\left(3-\frac{5}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{3-\frac{5}{n^2}}$$

So far we have only carried along the "limit" notation. We now observe that because we can make the values n^2 as large as we like by taking n sufficiently large, we can make $1/n^2$ and $5/n^2$ as close as we like to 0 by taking n sufficiently large. It follows that we can make the numerator of the main fraction as close as we like to 2 and the denominator of the main fraction as close as we like to 3 by taking n sufficiently large. In symbols,

$$\lim_{n\to\infty} \frac{2n^2+1}{3n^2-5} = \lim_{n\to\infty} \frac{2+\dfrac{1}{n^2}}{3-\dfrac{5}{n^2}} = \frac{2}{3}$$

Now work Problem 7 ◁

PROBLEMS 4.7

In Problems 1–4, find the present value of the given perpetuity.

1. \$60 per month at the rate of 1.5% monthly.

2. \$5000 per month at the rate of 0.5% monthly.

3. \$60,000 per year at the rate of 8% yearly.

4. \$4000 per year at the rate of 10% yearly.

5. Funding a Prize The Commerce Society would like to endow an annual prize of \$120 to the student who is deemed to have exhibited the most class spirit. The society is confident that it can invest indefinitely at an interest rate of at least 2.5% a year. How much does the society need to endow its prize?

In Problems 6–9, find the limit.

6. $\lim_{n\to\infty} \frac{n+5}{3n^2+2n-7}$

7. $\lim_{n\to\infty} \frac{n^2+3n-6}{n^2+4}$

8. $\lim_{k\to\infty} \left(\frac{k+1}{k}\right)^{2k}$

9. $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$

Chapter 4 Review

Key Terms

Section 4.1 **Summation Notation and Sequences**
arithmetic sequence, geometric sequence
summation notation, sum of a geometric sequence
sum of an arithmetic sequence

Section 4.2 **Simple and Compound Interest**
annual percentage yield (APY)
compound amount, effective rate, future value, principal

Section 4.3 **Present Value**
cash flows, equation of value
net present value, present value

Section 4.4 **Interest Compounded Continuously**
compounded continuously

Section 4.5 **Annuities**
annuity, annuity due
future value of an annuity $s_{\overline{n}|r}$
Murabahah, ordinary annuity, sinking fund

Section 4.6 **Amortization of Loans**
amortization schedules, amortizing, finance charge

Section 4.7 **Perpetuities**
perpetuity

Summary

Summation notation provides a compact and precise way of writing sums that have many terms. The basic equations of summation notation are just restatements of the properties of addition. Certain particular sums, such as $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^2$, are memorable and useful.

Both arithmetic sequences and geometric sequences arise in applications, particularly in business applications. Sums of sequences, particularly those of geometric sequences, are important in our study of the mathematics of finance.

The concept of compound interest lies at the heart of any discussion dealing with the time value of money—that is, the present value of money due in the future or the future value of money presently invested. Under compound interest, interest is converted into principal and earns interest itself. The basic compound-interest formulas are

$$S = P(1+r)^n \qquad \text{future value}$$
$$P = S(1+r)^{-n} \qquad \text{present value}$$

where S = compound amount (future value)

P = principal (present value)

r = periodic rate

n = number of interest periods.

Interest rates are usually quoted as an annual rate called the nominal rate. The periodic rate is obtained by dividing the nominal rate by the number of interest periods each year. The effective rate is the annual simple-interest rate which is equivalent to the nominal rate of r compounded n times a year and is given by

$$r_e = \left(1 + \frac{r}{n}\right)^n - 1 \qquad \text{effective rate}$$

Effective rates are used to compare different interest rates.

If interest is compounded continuously, then

$$S = Pe^{rt} \qquad \text{future value}$$
$$P = Se^{-rt} \qquad \text{present value}$$

where S = compound amount (future value)

P = principal (present value)

r = annual rate

t = number of years.

and the effective rate is given by

$$r_e = e^r - 1 \qquad \text{effective rate}$$

An annuity is a sequence of payments made at fixed periods of time over some interval. The mathematical basis for formulas dealing with annuities is the notion of the sum of a geometric sequence—that is,

$$s = \sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r} \qquad \text{sum of geometric sequence}$$

where s = sum
a = first term
r = common ratio
n = number of terms.

An ordinary annuity is an annuity in which each payment is made at the *end* of a payment period, whereas an annuity due is an annuity in which each payment is made at the *beginning* of a payment period. The basic formulas dealing with ordinary annuities are

$$A = R \cdot \frac{1-(1+r)^{-n}}{r} = Ra_{\overline{n}|r} \qquad \text{present value}$$

$$S = R \cdot \frac{(1+r)^n - 1}{r} = Rs_{\overline{n}|r} \qquad \text{future value}$$

where A = present value of annuity
S = amount (future value) of annuity
R = amount of each payment
n = number of payment periods
r = periodic rate.

For an annuity due, the corresponding formulas are

$$A = R(1 + a_{\overline{n-1}|r}) \qquad \text{present value}$$

$$S = R(s_{\overline{n+1}|r} - 1) \qquad \text{future value}$$

Banks in many Arab countries use Murabahah contracts to lend money to customers in accordance with Islamic principles. When a customer wants to buy an asset, the bank purchases the asset and sells it back to the customer under a Murabahah, which stipulates the cost incurred by the bank in buying the asset, along with an agreed-upon rate of profit for the bank; these numbers are entered into a formula to calculate the amount of periodic payments to be made by the customer.

A loan, such as a mortgage, is amortized when part of each installment payment is used to pay interest and the remaining part is used to reduce the principal. A complete analysis of each payment is given in an amortization schedule. The following formulas deal with amortizing a loan of A dollars, at the periodic rate of r, by n equal payments of R dollars each and such that a payment is made at the end of each period:

Periodic payment:

$$R = \frac{A}{a_{\overline{n}|r}} = A \cdot \frac{r}{1-(1+r)^{-n}}$$

Principal outstanding at beginning of kth period:

$$Ra_{\overline{n-k+1}|r} = R \cdot \frac{1-(1+r)^{-n+k-1}}{r}$$

Interest in kth payment:

$$Rra_{\overline{n-k+1}|r}$$

Principal contained in kth payment:

$$R(1 - ra_{\overline{n-k+1}|r})$$

Total interest paid:

$$R(n - a_{\overline{n}|r}) = nR - A$$

A perpetuity is an infinite sequence of payments made at fixed periods of time. The mathematical basis for the formula dealing with a perpetuity is the notion of the sum of an infinite geometric sequence—that is,

$$s = \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} \qquad \text{sum of infinite geometric sequence}$$

where s = sum
a = first term
r = common ratio with $|r| < 1$.

The basic formula dealing with perpetuities is

$$A = \frac{R}{r} \qquad \text{present value}$$

where A = present value of perpetuity
R = amount of each payment
r = periodic rate.

An infinite sum is defined as the limit of the sequence of partial sums.

Review Problems

1. Evaluate $\sum_{k=1}^{8}(k+3)^3$ by first cubing the binomial and then using Equations (10), (13), (14), and (15) of Section 4.1.

2. Evaluate $\sum_{i=4}^{11} i^3$ by using $\sum_{i=1}^{11} i^3 - \sum_{i=1}^{3} i^3$. Explain why this works quoting any equations from Section 4.1 that are used. Explain why the answer is necessarily the same as that in Problem 1.

3. Write the first five terms of the arithmetic sequence with first term 32 and common difference 3.

4. Write the first five terms of the geometric sequence with first term 100 and common ratio 1.02.

5. Find the sum of the first five terms of the arithmetic sequence with first term 32 and common difference 3.

6. Find the sum of the first five terms of the geometric sequence with first term 100 and common ratio 1.02.

7. Find the number of interest periods that it takes for a principal to double when the interest rate is r per period.

8. Find the effective rate that corresponds to a nominal rate of 5% compounded monthly.

9. An investor has a choice of investing a sum of money at either 8.5% compounded annually or 8.2% compounded semiannually. Which is the better choice?

10. Cash Flows Find the net present value of the following cash flows, which can be purchased by an initial investment of \$7000:

Year	Cash Flow
2	\$3400
4	3500

Assume that interest is at 7% compounded semiannually.

11. A debt of \$1500 due in five years and \$2000 due in seven years is to be repaid by a payment of \$2000 now and a second payment at the end of three years. How much should the second payment be if interest is at 3% compounded annually?

12. Find the present value of an annuity of \$250 at the end of each month for four years if interest is at 6% compounded monthly.

13. For an annuity of \$200 at the end of every six months for $6\frac{1}{2}$ years, find (**a**) the present value and (**b**) the future value at an interest rate of 8% compounded semiannually.

14. Find the amount of an annuity due which consists of 13 yearly payments of \$150, provided that the interest rate is 4% compounded annually.

15. Suppose \$200 is initially placed in a savings account and \$200 is deposited at the end of every month for the next year. If interest is at 8% compounded monthly, how much is in the account at the end of the year?

16. A savings account pays interest at the rate of 2% compounded semiannually. What amount must be deposited now so that \$350 can be withdrawn at the end of every six months for the next 15 years?

17. Sinking Fund A company borrows \$5000 on which it will pay interest at the end of each year at the annual rate of 11%. In addition, a sinking fund is set up so that the \$5000 can be repaid at the end of five years. Equal payments are placed in the fund at the end of each year, and the fund earns interest at the effective rate of 6%. Find the annual payment in the sinking fund.

18. Car Loan A debtor is to amortize a \$7000 car loan by making equal payments at the end of each month for 36 months. If interest is at 4% compounded monthly, find (**a**) the amount of each payment and (**b**) the finance charge.

19. A person has debts of \$500 due in three years with interest at 5% compounded annually and \$500 due in four years with interest at 6% compounded semiannually. The debtor wants to pay off these debts by making two payments: the first payment now, and the second, which is double the first payment, at the end of the third year. If money is worth 7% compounded annually, how much is the first payment?

20. Construct an amortization schedule for a loan of \$3500 repaid by three monthly payments with interest at 16.5% compounded monthly.

21. Construct an amortization schedule for a loan of \$15,000 repaid by five monthly payments with interest at 9% compounded monthly.

22. Find the present value of an ordinary annuity of \$460 every month for nine years at the rate of 6% compounded monthly.

23. Auto Loan Determine the finance charge for a 48-month auto loan of \$11,000 with monthly payments at the rate of 5.5% compounded monthly.

Chapter Test

1. Express the following sum in summation notation: $11 + 15 + 19 + 23 + \cdots + 71$

2. Evaluate the sum $\sum_{k=1}^{50}(k+50)^2$.

3. Evaluate $\sum_{k=1}^{n}\left\{\left[5-\left(\frac{3}{n}\cdot k\right)^2\right]\frac{3}{n}\right\}$.

4. Determine whether the given sequences are equal to each other: $(j^3 - 9j^2 + 27j - 27)_{j=1}^{\infty}$ and $((k-3)^3)_{k=1}^{\infty}$.

5. Write down the first five terms of the geometric sequence with $a = 100$, $r = 1.05$.

6. Find the following infinite sum if possible, or state why this cannot be done: $\sum_{k=1}^{\infty}\frac{2}{3}(1.5)^{k-1}$

7. Future Value Amani has just turned seven years old. She would like to save some money each month, starting next month, so that on her 21st birthday she will have \$1000 in her bank account. Marge told her that with current interest rates her kth deposit will be worth, on her 21st birthday, $(1.004)^{168-k}$ times the deposited amount. Amani wants to deposit the same amount each month. Write a formula for the amount Amani needs to deposit each month to meet her goal. Use your calculator to evaluate the required amount.

8. Find (**i**) the compound interest (rounded to two decimal places) and (**ii**) the effective rate (to three decimal places) if \$1000 is invested for five years at an annual rate of 7% compounded

(a) quarterly **(b)** monthly
(c) weekly **(d)** daily.

9. A \$6000 certificate of deposit is purchased for \$6000 and is held for seven years. If the certificate earns an effective rate of 8%, what is it worth at the end of that period?

10. How long will it take for \$100 to amount to \$1000 if invested at 6% compounded monthly? Express the answer in years, rounded to two decimal places.

11. Inflation If the rate of inflation for certain goods is $7\frac{1}{4}$% compounded daily, how many years will it take for the average price of such goods to double?

12. Find the present value of a future payment of \$9000 due in $5\frac{1}{2}$ years at 8% compounded quarterly.

13. A bank account pays 5.3% annual interest, compounded monthly. How much must be deposited now so that the account contains exactly \$12,000 at the end of one year?

14. A debt of \$3500 due in four years and \$5000 due in six years is to be repaid by a single payment of \$1500 now and three equal payments that are due each consecutive year from now. If the interest rate is 7% compounded annually, how much are each of the equal payments?

15. Investment If \$100 is deposited in a savings account that earns interest at an annual rate of $4\frac{1}{2}$% compounded continuously, what is the value of the account at the end of two years?

16. Investment Presently, Rachida and Adil have \$50,000 to invest for 18 months. They have two options open to them: **(a)** Invest the money in a certificate paying interest at the nominal rate of 5% compounded quarterly. **(b)** Invest the money in a savings account earning interest at the annual rate of 4.5% compounded continuously. How much money will they have in 18 months with each option?

17. If interest is compounded continuously at an annual rate of 3%, in how many years will a principal double? Give the answer correct to two decimal places.

18. Find the present value of an (ordinary) annuity of \$1000 every six months for four years at the rate of 10%. compounded semiannually.

19. Equipment Purchase A machine is purchased for \$3000 down and payments of \$250 at the end of every six months for six years. If interest is at 8% compounded semiannually, find the corresponding cash price of the machine.

20. Find $s_{\overline{60}|0.017}$ to five decimal places.

21. Find $250a_{\overline{180}|0.0235}$ to two decimal places.

22. A person wishes to make a three-year loan and can afford payments of \$50 at the end of each month. If interest is at 12% compounded monthly, how much can the person afford to borrow?

23. A person borrows \$2000 and will pay off the loan by equal payments at the end of each month for five years. If interest is at the rate of 16.8% compounded monthly, how much is each payment?

24. Retirement Planning Starting a year from now and making 10 yearly payments, Amir would like to put into a retirement account enough money so that, starting 11 years from now, he can withdraw \$30,000 per year until he dies. Amir is confident that he can earn 8% per year on his money for the next 10 years, but he is only assuming that he will be able to get 5% per year after that. **(a)** How much does Amir need to pay each year for the first 10 years in order to make the planned withdrawals? **(b)** Amir's will states that, upon his death, any money left in his retirement account is to be donated to the Al Azhar University. If he dies immediately after receiving his 17th payment, how much will the Al Azhar University inherit?

EXPLORE & EXTEND Treasury Securities

After completing his studies in the United States, Anwar Yassine wants to make an investment there before he leaves. He has been told that the safest single type of investment is in securities issued by the U.S. Treasury. These pay fixed returns on a predetermined schedule, which can span as little as three months or as much as thirty years. The finish date is called the date of maturity.

Although Treasury securities are initially sold by the government, they trade on the open market. Because the prices are free to float up and down, securities' rates of return can change over time. Consider, for example, a six-month Treasury bill, or T-bill, bearing a \$10,000 face value and purchased on the date of issue for \$9832.84. T-bills pay