GLOBAL EDITION

THOMAS' CALCULUS

FIFTEENTH EDITION IN SI UNITS

Hass • Heil • Bogacki • Weir

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Proof that $\ln x^r = r \ln x$ (assuming r rational) We use the same-derivative argument again. For all positive values of x,

$$\begin{aligned}\frac{d}{dx}\ln x^r &= \frac{1}{x^r}\frac{d}{dx}(x^r) && \text{Eq. (2) with } u = x^r\\ &= \frac{1}{x^r} r x^{r-1} && \text{General Power Rule for derivatives, } r \text{ rational}\\ &= r\cdot\frac{1}{x} = \frac{d}{dx}(r\ln x).\end{aligned}$$

Since $\ln x^r$ and $r \ln x$ have the same derivative,

$$\ln x^r = r \ln x + C$$

for some constant C. Taking x to be 1 identifies C as zero, and we're done. (Exercise 48 in Section 3.7 indicates a proof of the General Power Rule for derivatives when r is rational.) ■

You are asked to prove Rule 2 in Exercise 90. Rule 3 is a special case of Rule 2, obtained by setting $b = 1$ and noting that $\ln 1 = 0$. This covers all cases of Theorem 2.

We have not yet proved Rule 4 for r irrational; however, the rule does hold for all r, rational or irrational. We will show this in the next section after we define exponential functions and irrational exponents.

The Graph and Range of $\ln x$

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down. (See Figure 7.9a.)

We can estimate the value of $\ln 2$ by considering the area under the graph of $y = 1/x$ and above the interval $[1, 2]$. In Figure 7.9b a rectangle of height $1/2$ over the interval $[1, 2]$ fits under the graph. Therefore, the area under the graph, which is $\ln 2$, is greater than the area of the rectangle, which is $1/2$. So $\ln 2 > 1/2$. Knowing this we have

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}.$$

This result shows that $\ln(2^n) \to \infty$ as $n \to \infty$. Since $\ln x$ is an increasing function, it follows that

$$\lim_{x\to\infty} \ln x = \infty. \qquad \text{ln } x \text{ is increasing and not bounded above.}$$

We also have

$$\lim_{x\to 0^+} \ln x = \lim_{t\to\infty} \ln\frac{1}{t} = \lim_{t\to\infty}(-\ln t) = -\infty. \qquad x = 1/t$$

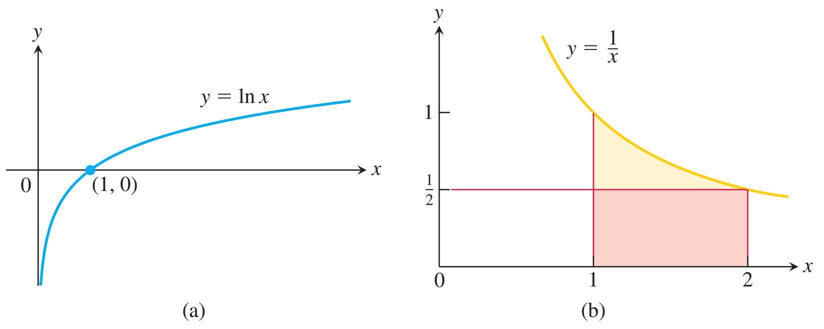

FIGURE 7.9 (a) The graph of the natural logarithm. (b) The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

We defined $\ln x$ for $x > 0$, so the domain of $\ln x$ is the set of positive real numbers. The above discussion and the Intermediate Value Theorem show that its range is the entire real line, giving the familiar graph of $y = \ln x$ shown in Figure 7.9a.

The Integral $\int 1/x\, dx$

Equation (3) leads to the following integral formula:

$$\int \frac{1}{x}\,dx = \ln|x| + C. \tag{4}$$

If u is a differentiable function that is never zero, then

$$\int \frac{1}{u}\,du = \ln|u| + C. \tag{5}$$

Equation (5) applies anywhere on the domain of $1/u$, which is the set of points where $u \neq 0$. It says that integrals that have the form $\int \frac{du}{u}$ lead to logarithms. Whenever $u = f(x)$ is a differentiable function that is never zero, we have that $du = f'(x)\,dx$ and

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

EXAMPLE 3 We rewrite an integral so that it has the form $\int \frac{du}{u}$.

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3 + 2\sin\theta}\,d\theta &= \int_1^5 \frac{2}{u}\,du \\ &= 2\ln|u|\Big]_1^5 \\ &= 2\ln|5| - 2\ln|1| = 2\ln 5\end{aligned}$$

$u = 3 + 2\sin\theta,\quad du = 2\cos\theta\,d\theta,$
$u(-\pi/2) = 1,\quad u(\pi/2) = 5$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (5) applies. ■

The Integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

Equation (3) tells us how to integrate these trigonometric functions.

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} \\ &= -\ln|u| + C = -\ln|\cos x| + C \\ &= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C.\end{aligned}$$

$u = \cos x$
$du = -\sin x\,dx$

Reciprocal Rule

For the cotangent,

$$\begin{aligned}\int \cot x\,dx &= \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} \\ &= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.\end{aligned}$$

$u = \sin x,$
$du = \cos x\,dx$

To integrate $\sec x$, we multiply and divide by $(\sec x + \tan x)$ as an algebraic form of 1.

$$\int \sec x\, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

$u = \sec x + \tan x$,
$du = (\sec x \tan x + \sec^2 x)\, dx$

For $\csc x$, we multiply and divide by $(\csc x + \cot x)$.

$$\int \csc x\, dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)}\, dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$$

$u = \csc x + \cot x$,
$du = (-\csc x \cot x - \csc^2 x)\, dx$

In summary, we have the following results.

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\, dx = \ln|\sin x| + C \qquad \int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

EXAMPLE 4

$$\int_0^{\pi/6} \tan 2x\, dx = \int_0^{\pi/3} \tan u \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\, du$$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$,
$dx = du/2$,
$u(0) = 0$,
$u(\pi/6) = \pi/3$

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called **logarithmic differentiation**, is illustrated in the next example.

EXAMPLE 5 Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms:

$$\ln y = \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}$$

$$= \ln\left((x^2 + 1)(x + 3)^{1/2}\right) - \ln(x - 1) \qquad \text{Quotient Rule}$$

$$= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) \qquad \text{Product Rule}$$

$$= \ln(x^2 + 1) + \frac{1}{2}\ln(x + 3) - \ln(x - 1). \qquad \text{Power Rule}$$

We then take derivatives of both sides with respect to x, using Equation (2):

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$ ■

The computation in Example 5 would be much longer if we used the product, quotient, and power rules.

EXERCISES 7.2

Using the Algebraic Properties—Theorem 2

1. Express the following logarithms in terms of ln 2 and ln 3.

a. $\ln 0.75$ **b.** $\ln(4/9)$ **c.** $\ln(1/2)$

d. $\ln\sqrt[3]{9}$ **e.** $\ln 3\sqrt{2}$ **f.** $\ln\sqrt{13.5}$

2. Express the following logarithms in terms of ln 5 and ln 7.

a. $\ln(1/125)$ **b.** $\ln 9.8$ **c.** $\ln 7\sqrt{7}$

d. $\ln 1225$ **e.** $\ln 0.056$

f. $(\ln 35 + \ln(1/7))/(\ln 25)$

Use the properties of logarithms to write the expressions in Exercises 3 and 4 as a single term.

3. a. $\ln\sin\theta - \ln\left(\frac{\sin\theta}{5}\right)$ **b.** $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$

c. $\frac{1}{2}\ln(4t^4) - \ln 2$

4. a. $\ln\sec\theta + \ln\cos\theta$

b. $\ln(8x + 4) - 2\ln 2$

c. $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1)$

In Exercises 5 and 6, solve for t.

5. $\ln(t - 1) + \ln(t - 4) = \ln(-t)$

6. $\ln(t - 2) = \ln 8 - \ln t$

Finding Derivatives

In Exercises 7–38, find the derivative of y with respect to x, t, or θ, as appropriate.

7. $y = \ln 3x$

8. $y = \ln kx$, k constant

9. $y = \ln(t^2)$

10. $y = \ln(t^{3/2})$

11. $y = \ln\frac{3}{x}$

12. $y = \ln\frac{10}{x}$

13. $y = \ln(\theta + 1)$

14. $y = \ln(2\theta + 2)$

15. $y = \ln x^3$

16. $y = (\ln x)^3$

17. $y = t(\ln t)^2$

18. $y = t\sqrt{\ln t}$

19. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$

20. $y = (x^2\ln x)^4$

21. $y = \frac{\ln t}{t}$

22. $y = \frac{1 + \ln t}{t}$

23. $y = \frac{\ln x}{1 + \ln x}$

24. $y = \frac{x\ln x}{1 + \ln x}$

25. $y = \ln(\ln x)$

26. $y = \ln(\ln(\ln x))$

27. $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$

28. $y = \ln(\sec\theta + \tan\theta)$

29. $y = \ln\frac{1}{x\sqrt{x + 1}}$

30. $y = \frac{1}{2}\ln\frac{1 + x}{1 - x}$

31. $y = \frac{1 + \ln t}{1 - \ln t}$

32. $y = \sqrt{\ln\sqrt{t}}$

33. $y = \ln(\sec(\ln\theta))$

34. $y = \ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1 + 2\ln\theta}\right)$

35. $y = \ln\left(\frac{(x^2 + 1)^5}{\sqrt{1 - x}}\right)$

36. $y = \ln\sqrt{\frac{(x + 1)^5}{(x + 2)^{20}}}$

37. $y = \int_{x^2/2}^{x^2}\ln\sqrt{t}\,dt$

38. $y = \int_{\sqrt{x}}^{\sqrt[3]{x}}\ln t\,dt$

Evaluating Integrals

Evaluate the integrals in Exercises 39–56.

39. $\int_{-3}^{-2}\frac{dx}{x}$

40. $\int_{-1}^{0}\frac{3\,dx}{3x - 2}$

41. $\int\frac{2y\,dy}{y^2 - 25}$

42. $\int\frac{8r\,dr}{4r^2 - 5}$

43. $\int_0^{\pi}\frac{\sin t}{2 - \cos t}\,dt$

44. $\int_0^{\pi/3}\frac{4\sin\theta}{1 - 4\cos\theta}\,d\theta$

45. $\int_1^2\frac{2\ln x}{x}\,dx$

46. $\int_2^4\frac{dx}{x\ln x}$

47. $\int_2^4 \frac{dx}{x(\ln x)^2}$

48. $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

49. $\int \frac{3\sec^2 t}{6 + 3\tan t}\,dt$

50. $\int \frac{\sec y \tan y}{2 + \sec y}\,dy$

51. $\int_0^{\pi/2} \tan \frac{x}{2}\,dx$

52. $\int_{\pi/4}^{\pi/2} \cot t\,dt$

53. $\int_{\pi/2}^{\pi} 2\cot \frac{\theta}{3}\,d\theta$

54. $\int_0^{\pi/12} 6\tan 3x\,dx$

55. $\int \frac{dx}{2\sqrt{x} + 2x}$

56. $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

Logarithmic Differentiation

In Exercises 57–70, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

57. $y = \sqrt{x(x+1)}$

58. $y = \sqrt{(x^2+1)(x-1)^2}$

59. $y = \sqrt{\frac{t}{t+1}}$

60. $y = \sqrt{\frac{1}{t(t+1)}}$

61. $y = \sqrt{\theta + 3}\sin\theta$

62. $y = (\tan\theta)\sqrt{2\theta + 1}$

63. $y = t(t+1)(t+2)$

64. $y = \frac{1}{t(t+1)(t+2)}$

65. $y = \frac{\theta + 5}{\theta\cos\theta}$

66. $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}$

67. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

68. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$

69. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$

70. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

Theory and Applications

71. Locate and identify the absolute extreme values of

a. $\ln(\cos x)$ on $[-\pi/4, \pi/3]$,

b. $\cos(\ln x)$ on $[1/2, 2]$.

72. a. Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.

b. Using part (a), show that $\ln x < x$ if $x > 1$.

73. Find the area between the curves $y = \ln x$ and $y = \ln 2x$ from $x = 1$ to $x = 5$.

74. Find the area between the curve $y = \tan x$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$.

In Exercises 75 and 76:

a. Find the open intervals on which the function is increasing and decreasing.

b. Identify the function's local and absolute extreme values, if any, saying where they occur.

75. $g(x) = x(\ln x)^2$

76. $g(x) = x^2 - 2x - 4\ln x$

77. The region in the first quadrant bounded by the coordinate axes, the line $y = 3$, and the curve $x = 2/\sqrt{y+1}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

78. The region between the curve $y = \sqrt{\cot x}$ and the x-axis from $x = \pi/6$ to $x = \pi/2$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

79. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

80. In Section 6.2, Exercise 6, we revolved about the y-axis the region between the curve $y = 9x/\sqrt{x^3 + 9}$ and the x-axis from $x = 0$ to $x = 3$ to generate a solid of volume 36π. What volume do you get if you revolve the region about the x-axis instead? (See Section 6.2, Exercise 6, for a graph.)

81. Find the lengths of the following curves.

a. $y = (x^2/8) - \ln x, \quad 4 \le x \le 8$

b. $x = (y/4)^2 - 2\ln(y/4), \quad 4 \le y \le 12$

82. Find a curve through the point (1, 0) whose length from $x = 1$ to $x = 2$ is

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^2}}\,dx.$$

T **83. a.** Find the centroid of the region between the curve $y = 1/x$ and the x-axis from $x = 1$ to $x = 2$. Give the coordinates to two decimal places.

b. Sketch the region and show the centroid in your sketch.

84. a. Find the center of mass of a thin plate of constant density covering the region between the curve $y = 1/\sqrt{x}$ and the x-axis from $x = 1$ to $x = 16$.

b. Find the center of mass if, instead of being constant, the density function is $\delta(x) = 4/\sqrt{x}$.

85. Use a derivative to show that $f(x) = \ln(x^3 - 1)$ is one-to-one.

86. Use a derivative to show that $g(x) = \sqrt{x^2 + \ln x}$ is one-to-one.

Solve the initial value problems in Exercises 87 and 88.

87. $\frac{dy}{dx} = 1 + \frac{1}{x}, \quad y(1) = 3$

88. $\frac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1$

T **89. The linearization of $\ln(1 + x)$ at $x = 0$** Instead of approximating $\ln x$ near $x = 1$, we approximate $\ln(1 + x)$ near $x = 0$. We get a simpler formula this way.

a. Derive the linearization $\ln(1 + x) \approx x$ at $x = 0$.

b. Estimate to five decimal places the error involved in replacing $\ln(1 + x)$ by x on the interval $[0, 0.1]$.

c. Graph $\ln(1 + x)$ and x together for $0 \le x \le 0.5$. Use different colors, if available. At what points does the approximation of $\ln(1 + x)$ seem best? Least good? By reading coordinates from the graphs, find as good an upper bound for the error as your grapher will allow.

90. Use the same-derivative argument, as was done to prove Rules 1 and 4 of Theorem 2, to prove the Quotient Rule property of logarithms.

T **91. a.** Graph $y = \sin x$ and the curves $y = \ln(a + \sin x)$ for $a = 2, 4, 8, 20$, and 50 together for $0 \le x \le 23$.

b. Why do the curves flatten as a increases? (*Hint:* Find an a-dependent upper bound for $|y'|$.)

T **92.** Does the graph of $y = \sqrt{x} - \ln x$, $x > 0$, have an inflection point? Try to answer this question **(a)** by graphing, **(b)** by using calculus.

7.3 Exponential Functions

Having developed the theory of the function $\ln x$, we introduce its inverse, the exponential function $\exp x = e^x$. We study its properties and compute its derivative and integral. We prove the power rule for derivatives involving general real exponents. Finally, we introduce general exponential functions, a^x, and general logarithmic functions, $\log_a x$.

The Inverse of $\ln x$ and the Number e

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.10,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x\to -\infty} \ln^{-1} x = 0.$$

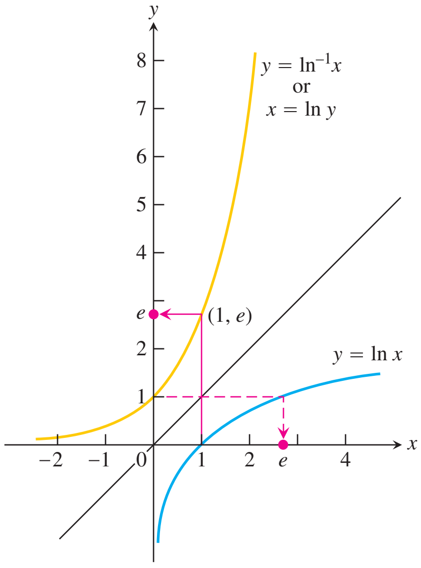

FIGURE 7.10 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

The inverse function $\ln^{-1} x$ is also denoted by $\exp x$. We have not yet established that $\exp x$ is an exponential function, only that $\exp x$ is the inverse of the function $\ln x$. We will now show that $\ln^{-1} x = \exp x$ is, in fact, the exponential function with base e.

The number e was defined to satisfy the equation $\ln(e) = 1$, so $e = \exp(1)$. We can raise the number e to a rational power r using algebra:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e}, \qquad e^{2/3} = \sqrt[3]{e^2},$$

and so on. Since e is positive, e^r is positive too, so we can take the logarithm of e^r. When we do, we find that for r rational

$$\ln e^r = r \ln e = r \cdot 1 = r. \qquad \text{Theorem 2, Rule 4}$$

Then applying the function $\ln^{-1}$ to both sides of the equation $\ln e^r = r$, we find that

$$e^r = \exp r \qquad \text{for } r \text{ rational.} \qquad \text{exp is } \ln^{-1}. \tag{1}$$