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THOMAS' CALCULUS

Early Transcendentals

FIFTEENTH EDITION IN SI UNITS

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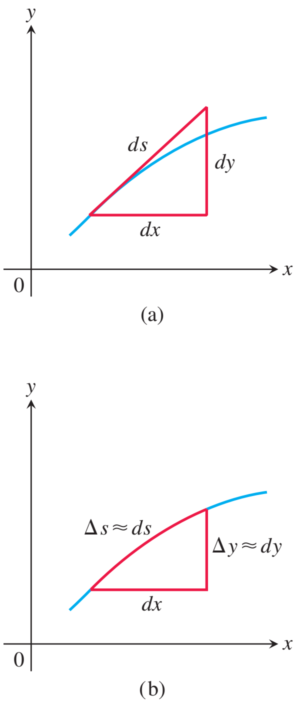


FIGURE 6.27 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

From Equation (3) and Figure 6.22, we see that this function $s(x)$ is continuous and measures the length along the curve $y = f(x)$ from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$ for each $x \in [a, b]$. The function s is called the **arc length function** for $y = f(x)$. From the Fundamental Theorem, the function s is differentiable on (a, b) and

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Then the differential of arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{6}$$

A useful way to remember Equation (6) is to write

$$ds = \sqrt{dx^2 + dy^2}, \tag{7}$$

which can be integrated between appropriate limits to give the total length of a curve. From this point of view, all the arc length formulas are simply different expressions for the equation $L = \int ds$. Figure 6.27a, which corresponds to Equation (7), can be thought of as a simplified approximation of Figure 6.27b. That is, ds is approximately equal to the exact arc length Δs.

EXAMPLE 5 Find the arc length function for the curve in Example 2, taking $A = (1, 13/12)$ as the starting point (see Figure 6.25).

Solution In the solution to Example 2, we found that

$$\sqrt{1 + [f'(x)]^2} = \frac{x^2}{4} + \frac{1}{x^2}.$$

Therefore the arc length function is given by

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + [f'(t)]^2}\, dt = \int_1^x \left(\frac{t^2}{4} + \frac{1}{t^2}\right) dt \\ &= \left[\frac{t^3}{12} - \frac{1}{t}\right]_1^x = \frac{x^3}{12} - \frac{1}{x} + \frac{11}{12}. \end{aligned}$$

To compute the arc length along the curve from $A = (1, 13/12)$ to $B = (4, 67/12)$, for instance, we simply calculate

$$s(4) = \frac{4^3}{12} - \frac{1}{4} + \frac{11}{12} = 6.$$

This is the same result we obtained in Example 2. ■

EXERCISES 6.3

Finding Lengths of Curves

Find the lengths of the curves in Exercises 1–16. If you have graphing software, you may want to graph these curves to see what they look like.

1. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
2. $y = x^{3/2}$ from $x = 0$ to $x = 4$
3. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
4. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
5. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
6. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
7. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \le x \le 8$
8. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$

9. $y = \ln x - \frac{x^2}{8}$ from $x = 1$ to $x = 2$

10. $y = \frac{x^2}{2} - \frac{\ln x}{4}$ from $x = 1$ to $x = 3$

11. $y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 3$

12. $y = \frac{x^5}{5} + \frac{1}{12x^3}, \quad \frac{1}{2} \le x \le 1$

13. $y = \frac{3}{2}x^{2/3} + 1, \quad \frac{1}{8} \le x \le 1$

14. $y = \frac{1}{2}(e^x + e^{-x}), \quad -1 \le x \le 1$

15. $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt, \quad -\pi/4 \le y \le \pi/4$

16. $y = \int_{-2}^x \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1$

T Finding Integrals for Lengths of Curves

In Exercises 17–24, do the following.

a. Set up an integral for the length of the curve.

b. Graph the curve to see what it looks like.

c. Use your grapher's or computer's integral evaluator to find the curve's length numerically.

17. $y = x^2, \quad -1 \le x \le 2$

18. $y = \tan x, \quad -\pi/3 \le x \le 0$

19. $x = \sin y, \quad 0 \le y \le \pi$

20. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$

21. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$

22. $y = \sin x - x\cos x, \quad 0 \le x \le \pi$

23. $y = \int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$

24. $x = \int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$

Theory and Examples

25. a. Find a curve with a positive derivative through the point $(1, 1)$ whose length integral (Equation 3) is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

b. How many such curves are there? Give reasons for your answer.

26. a. Find a curve with a positive derivative through the point $(0, 1)$ whose length integral (Equation 4) is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

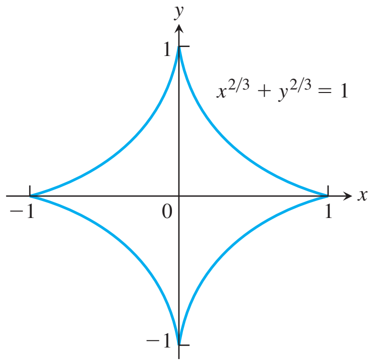
b. How many such curves are there? Give reasons for your answer.

27. Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt$$

from $x = 0$ to $x = \pi/4$.

28. The length of an astroid The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see the accompanying figure). Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y = (1 - x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.



29. Length of a line segment Use the arc length formula (Equation 3) to find the length of the line segment $y = 3 - 2x$, $0 \le x \le 2$. Check your answer by finding the length of the segment as the hypotenuse of a right triangle.

30. Circumference of a circle Set up an integral to find the circumference of a circle of radius r centered at the origin. You will learn how to evaluate the integral in Section 8.3.

31. If $9x^2 = y(y - 3)^2$, show that

$$ds^2 = \frac{(y + 1)^2}{4y}\, dy^2.$$

32. If $4x^2 - y^2 = 64$, show that

$$ds^2 = \frac{4}{y^2}(5x^2 - 16)\, dx^2.$$

33. Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

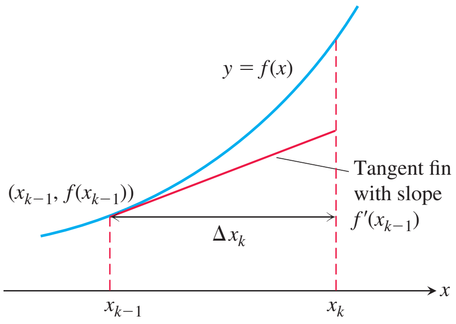
34. Using tangent fins to derive the length formula for curves Assume that f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.

a. Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\,\Delta x_k)^2}$.

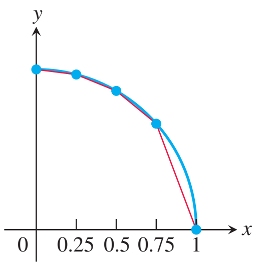
b. Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

which is the length L of the curve $y = f(x)$ from a to b.



35. Approximate the arc length of one-quarter of the unit circle (which is $\pi/2$) by computing the length of the polygonal approximation with $n = 4$ segments (see accompanying figure).



36. Distance between two points Assume that the two points (x_1, y_1) and (x_2, y_2) lie on the graph of the straight line $y = mx + b$. Use the arc length formula (Equation 3) to find the distance between the two points.

37. Find the arc length function for the graph of $f(x) = 2x^{3/2}$ using $(0, 0)$ as the starting point. What is the length of the curve from $(0, 0)$ to $(1, 2)$?

38. Find the arc length function for the curve in Exercise 8, using $(0, 1/4)$ as the starting point. What is the length of the curve from $(0, 1/4)$ to $(1, 59/24)$?

COMPUTER EXPLORATIONS

In Exercises 39–44, use a CAS to perform the following steps for the given graph of the function over the closed interval.

a. Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.22.)

b. Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

c. Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

39. $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

40. $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

41. $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

42. $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

43. $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$

44. $f(x) = x^3 - x^2, \quad -1 \le x \le 1$

6.4 Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The surface surrounds a volume of revolution, and many applications require that we know the area of the surface rather than the volume it encloses. In this section we define areas of surfaces of revolution. More general surfaces are treated in Chapter 15.

Defining Surface Area

If you revolve a region in the plane that is bounded by the graph of a function over an interval, it sweeps out a solid of revolution, as we saw earlier in the chapter. However, if you revolve only the bounding curve itself, it does not sweep out any interior volume but rather a surface that surrounds the solid and forms part of its boundary. Just as we were interested in defining and finding the length of a curve in the last section, we are now interested in defining and finding the area of a surface generated by revolving a curve about an axis.

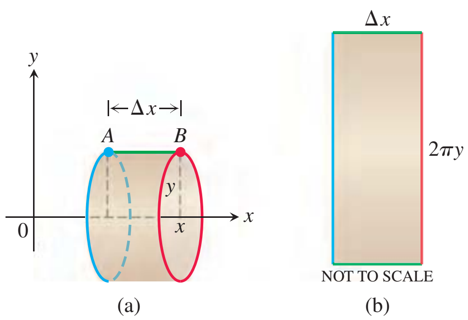


FIGURE 6.28 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y\,\Delta x$. (b) The cut and rolled-out cylindrical surface as a rectangle.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x-axis. If we rotate the horizontal line segment AB having length Δx about the x-axis (Figure 6.28a), we generate a cylinder with surface area $2\pi y\,\Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.28b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x-axis.

Suppose the line segment AB has length L and is slanted rather than horizontal. Now when AB is rotated about the x-axis, it generates a frustum of a cone (Figure 6.29a). From

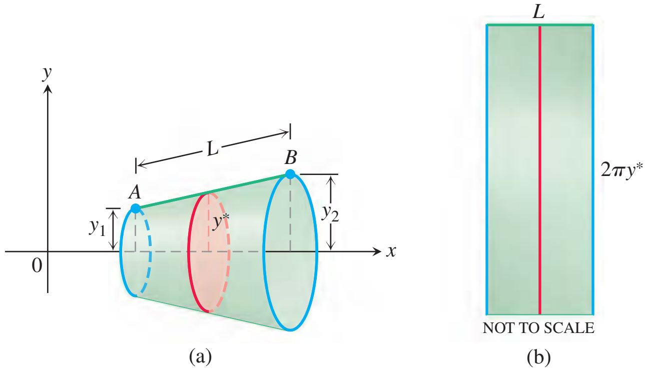


FIGURE 6.29 (a) The frustum of a cone generated by rotating the slanted line segment AB of length L about the x-axis has area $2\pi y^* L$. (b) The area of the rectangle for $y^* = \dfrac{y_1 + y_2}{2}$, the average height of AB above the x-axis.

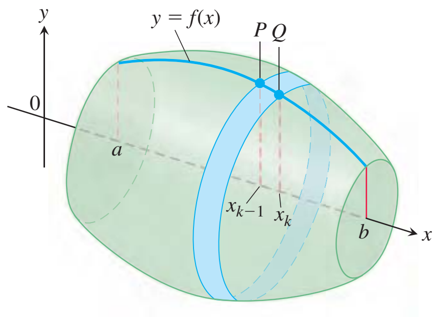


FIGURE 6.30 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

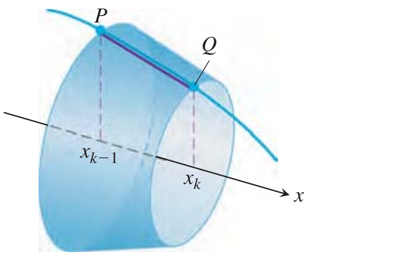


FIGURE 6.31 The line segment joining P and Q sweeps out a frustum of a cone.

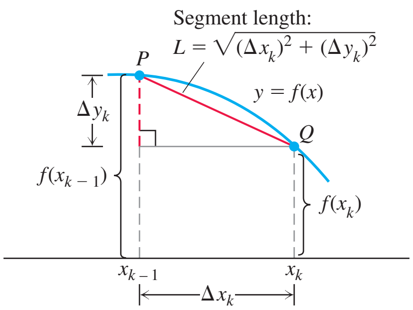


FIGURE 6.32 Dimensions associated with the arc and line segment PQ.

classical geometry, the surface area of this frustum is $2\pi y^* L$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment AB above the x-axis. This surface area is the same as that of a rectangle with side lengths L and $2\pi y^*$ (Figure 6.29b).

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the x-axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x)$, $a \le x \le b$, about the x-axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.30 shows a typical arc PQ and the band it sweeps out as part of the graph of f.

As the arc PQ revolves about the x-axis, the line segment joining P and Q sweeps out a frustum of a cone whose axis lies along the x-axis (Figure 6.31). The surface area of this frustum approximates the surface area of the band swept out by the arc PQ. The surface area of the frustum of the cone shown in Figure 6.31 is $2\pi y^* L$, where y^* is the average height of the line segment joining P and Q, and L is its length (just as before). Since $f \ge 0$, from Figure 6.32 we see that the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$\begin{aligned}\text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.\end{aligned}$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ, is approximated by the frustum area sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. To find the limit, we first need to find an appropriate substitution for Δy_k. If the function f is differentiable, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Figure 6.33). At this point,

$$\begin{aligned} f'(c_k) &= \frac{\Delta y_k}{\Delta x_k}, \\ \Delta y_k &= f'(c_k)\,\Delta x_k. \end{aligned}$$

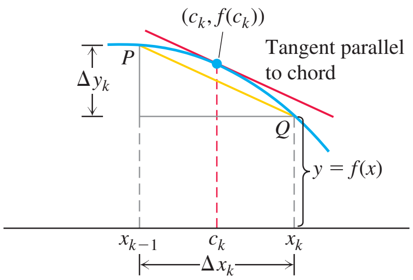


FIGURE 6.33 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ.

With this substitution for Δy_k, the sums in Equation (1) take the form

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k)\,\Delta x_k)^2}$$

$$= \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\,\Delta x_k. \qquad (2)$$

These sums are not the Riemann sums of any function because the points x_{k-1}, x_k, and c_k are not the same. However, the points x_{k-1}, x_k, and c_k are very close to each other, and so we expect (and it can be proved) that as the norm of the partition of $[a, b]$ goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of f from a to b.

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \qquad (3)$$

Note that the square root in Equation (3) is similar to the one that appears in the formula for the arc length of the generating curve in Equation (6) of Section 6.3.

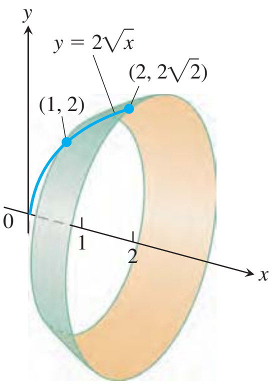


FIGURE 6.34 In Example 1 we calculate the area of this surface.

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis (Figure 6.34).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} = \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

With these substitutions, we have

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\,dx = 4\pi\int_1^2 \sqrt{x+1}\,dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$ ■

Revolution About the *y*-Axis

For revolution about the y-axis, we interchange x and y in Equation (3).

Surface Area for Revolution About the *y*-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

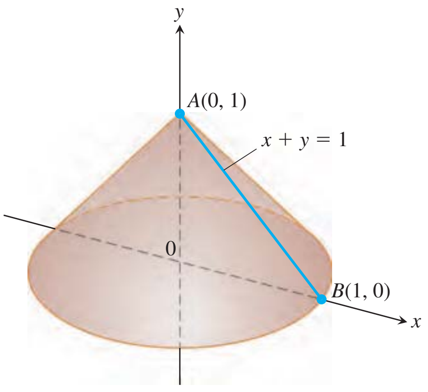


FIGURE 6.35 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

EXAMPLE 2 The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1 - y)\sqrt{2}\, dy$$

$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right) = \pi\sqrt{2}.$$

EXERCISES 6.4

Finding Integrals for Surface Area

In Exercises 1–8:

a. Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.

T **b.** Graph the curve to see what it looks like. If you can, graph the surface too.

T **c.** Use your utility's integral evaluator to find the surface's area numerically.

1. $y = \tan x, \quad 0 \leq x \leq \pi/4; \quad x\text{-axis}$

2. $y = x^2, \quad 0 \leq x \leq 2; \quad x\text{-axis}$

3. $xy = 1, \quad 1 \leq y \leq 2; \quad y\text{-axis}$

4. $x = \sin y, \quad 0 \leq y \leq \pi; \quad y\text{-axis}$

5. $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; x-axis

6. $y + 2\sqrt{y} = x, \quad 1 \leq y \leq 2; \quad y\text{-axis}$

7. $x = \int_0^y \tan t\, dt, \quad 0 \leq y \leq \pi/3; \quad y\text{-axis}$

8. $y = \int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \leq x \leq \sqrt{5}; \quad x\text{-axis}$

Finding Surface Area

9. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2$, $0 \leq x \leq 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2$, $0 \leq x \leq 4$, about the y-axis. Check your answer with the geometry formula given in Exercise 9.