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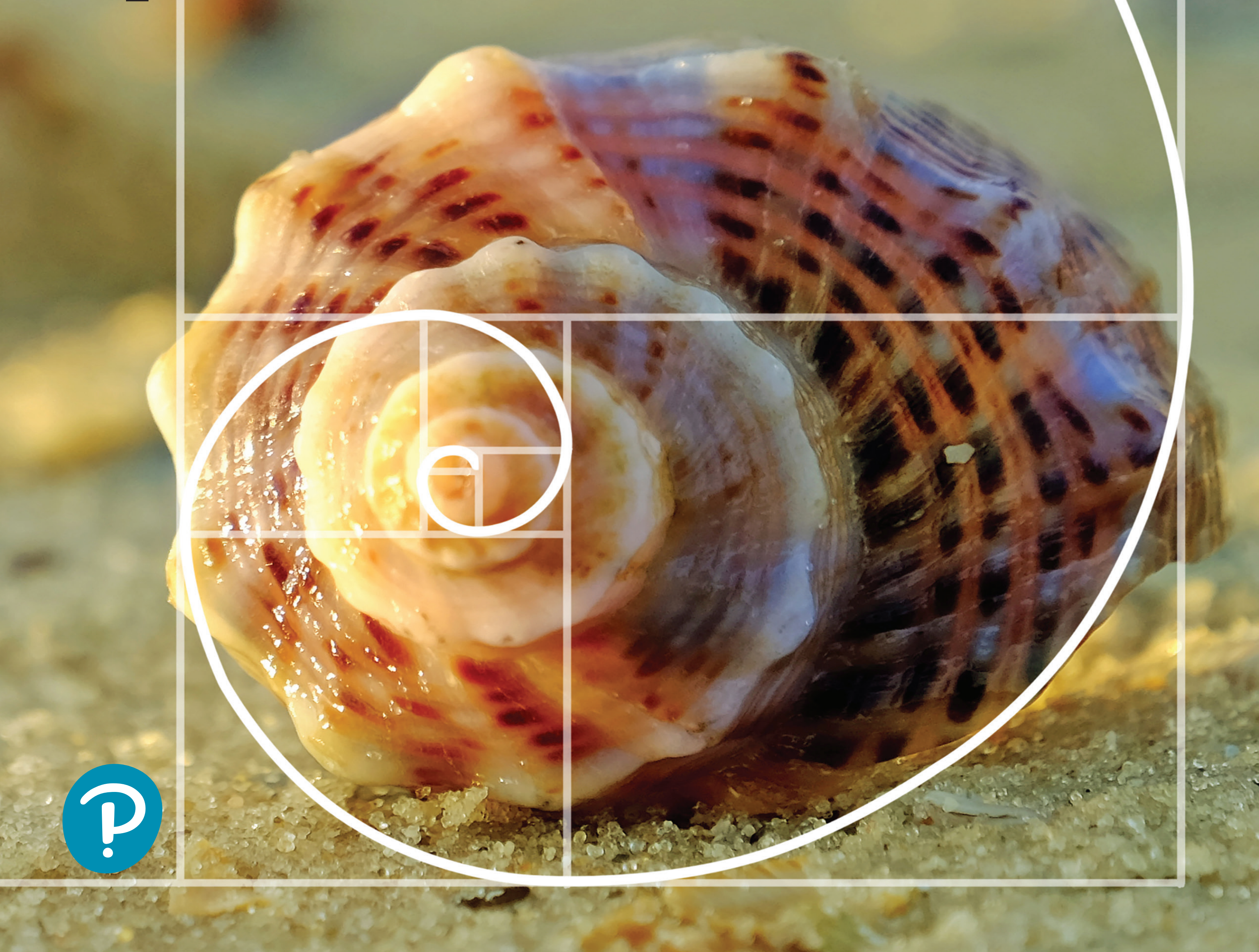
INTRODUCTORY MATHEMATICAL ANALYSIS

FOR BUSINESS, ECONOMICS, AND THE LIFE AND SOCIAL SCIENCES



INTRODUCTORY MATHEMATICAL ANALYSIS

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Review Problems

1. Find the number of interest periods that it takes for a principal to triple when the interest rate is r per period.

2. Find the effective rate that corresponds to a nominal rate of 16% compounded semiannually.

3. There are two investments in which one pays 3.6% compounded quarterly and the other pays 3.7% compounded annually on the same amount of investment. Which investment is better?

4. Cash Flows Find the net present value of a scheme that requires an initial investment of \$15,000 with the following cash flows:

Year	Cash Flow
3	1,600
4	2,400
6	2,700

Assume that interest is at 12% compounded quarterly.

5. A debt of \$1500 due in three years and \$3500 due in eight years is to be repaid by a single payment of \$1000 a year from now and a second payment at the end of five years. How much should the second payment be if interest is at 2% compounded annually?

6. Find the present value a bond which pays \$300 at the end of every six months for seven years with an interest of 3% compounded semiannually.

7. If a payment of \$1000 was being paid at the end of every four months for a 10-year annuity, find **(a)** the present value and **(b)** the future value at an interest rate of 14% compounded quarterly.

8. You buy an annuity that costs you \$250 per month for 4 years with an interest rate of 6% compounded monthly. Find amount due for the annuity.

9. Suppose you open a savings account with an initial amount of €100 and deposit €100 into the account at the end of every month for the next two years. How much is in the account at the end of the two years if interest is at 9% compounded monthly?

10. Find the amount that must be invested in an investment account that pays interest of 10% compounded quarterly so that \$500 can be withdrawn at the end of every quarter for the next nine years.

11. Sinking Fund A company deposits equal payments into a sinking fund at the end of each year to ensure the company can settle a debt of \$30,000 in seven years. Find the annual payment that the company must deposit into the sinking fund which will earn an interest at the effective rate of 6%.

12. Credit Card Cash Advance Due to an emergency, a person withdraws \$4000 using the credit card cash advance feature. The person will pay back the amount by making equal payments at the end of each month for 24 months. If interest is at 24% compounded monthly, find **(a)** the amount of each payment and **(b)** the finance charge.

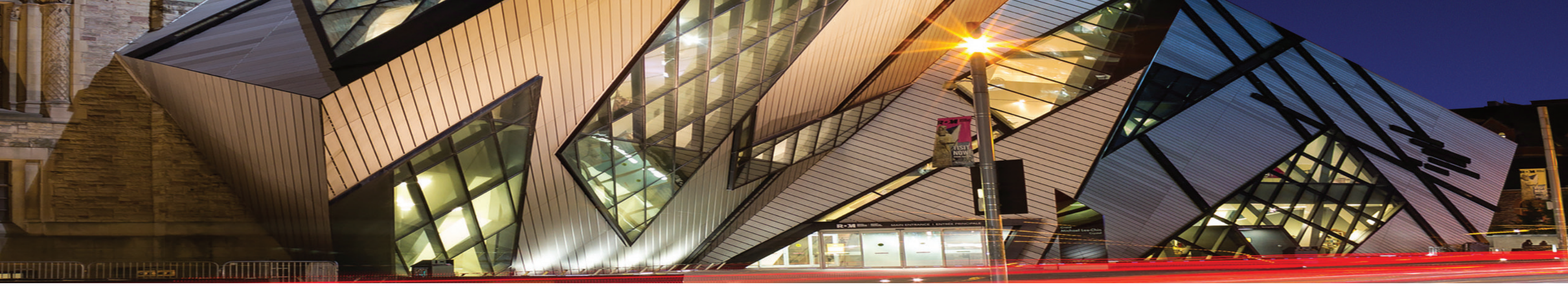
13. A person has debts of \$700 due in five years with interest at 3% compounded annually and \$850 due in seven years with interest at 8% compounded quarterly. The debtor wants to pay off these debts by making two payments: the first payment now, and the second, which is triple the first payment, at the end of the fourth year. If money is worth 6% compounded annually, how much is the first payment?

14. Construct an amortization schedule for a loan of \$7,200 repaid by three semiannually payments with interest at 7% compounded semiannually.

15. Construct an amortization schedule for a loan of \$43,000 repaid by five quarterly payments with interest at 16% compounded quarterly.

16. Find the present value of an ordinary annuity of \$120 every month for ten years at the rate of 3% compounded monthly.

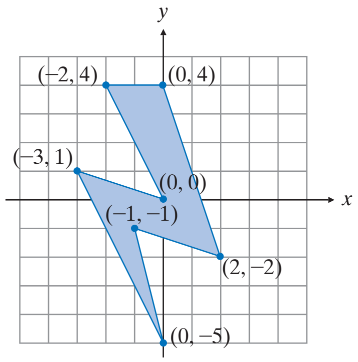
17. Housing Loan Determine the finance charge for a 360-month housing loan of \$360,000 with monthly payments at the rate of 3.15% compounded monthly.



6 Matrix Algebra

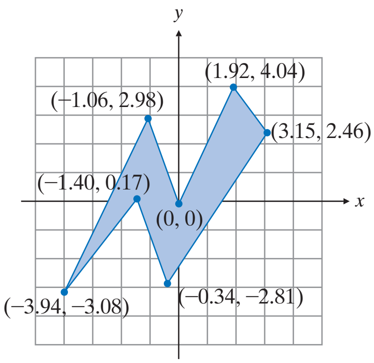
Matrices, the subject of this chapter, are simply arrays of numbers. Matrices and matrix algebra have potential application whenever numerical information can be meaningfully arranged into rectangular blocks.

One area of application for matrix algebra is computer graphics. An object in a coordinate system can be represented by a matrix that contains the coordinates of each corner. For example, we might set up a connect-the-dots scheme in which the lightning bolt shown is represented by the matrix to its right.



$$\begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array}$$
$$\begin{bmatrix} 0 & 0 \\ -2 & 4 \\ 0 & 4 \\ 2 & -2 \\ -1 & -1 \\ 0 & -5 \\ -3 & 1 \end{bmatrix}$$

Computer graphics often show objects rotating in space. Computationally, rotation is effected by matrix multiplication. The lightning bolt is rotated clockwise 52 degrees about the origin by matrix multiplication, involving a matrix whose entries are functions t_{11}, t_{12}, t_{21}, and t_{22} of the rotation angle (with $t_{11} = t_{22}$ and $t_{12} = -t_{21}$):



$$\begin{bmatrix} 0 & 0 \\ -2 & 4 \\ 0 & 4 \\ 2 & -2 \\ -1 & -1 \\ 0 & -5 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} t_{11}(52^\circ) & t_{12}(52^\circ) \\ t_{21}(52^\circ) & t_{22}(52^\circ) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1.92 & 4.04 \\ 3.15 & 2.46 \\ -0.34 & -2.81 \\ -1.40 & 0.17 \\ -3.94 & -3.08 \\ -1.06 & 2.98 \end{bmatrix}$$

Objective

To introduce the concept of a matrix and to consider special types of matrices.

6.1 Matrices

Finding ways to describe many situations in mathematics and economics leads to the study of rectangular arrays of numbers. Consider, for example, the system of linear equations

$$\begin{cases} 3x + 4y + 3z = 0 \\ 2x + y - z = 0 \\ 9x - 6y + 2z = 0 \end{cases}$$

If we are organized with our notation, keeping the x's in the first column, the y's in the second column, and so on, then the features that characterize this system are the numerical coefficients in the equations, together with their relative positions. For this reason, the system can be described by the rectangular arrays

$$\begin{bmatrix} 3 & 4 & 3 \\ 2 & 1 & -1 \\ 9 & -6 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

one for each *side* of the equations, each being called a *matrix* (plural: *matrices*, pronounced may′-tri-sees). We consider such rectangular arrays to be objects in themselves, and our custom, as just shown, will be to enclose them by brackets. Parentheses are also commonly used. In symbolically representing matrices, we use capital letters such as A, B, C, and so on.

Vertical bars, | |, around a rectangular array do *not* mean the same thing as brackets or parentheses.

In economics it is often convenient to use matrices in formulating problems and displaying data. For example, a manufacturer who produces products X, Y, and Z could represent the units of labor and material involved in one week's production of these items as in Table 6.1. More simply, the data can be represented by the matrix

Table 6.1

	Product		
	X	Y	Z
Labor	10	12	16
Material	5	9	7

$$A = \begin{bmatrix} 10 & 12 & 16 \\ 5 & 9 & 7 \end{bmatrix}$$

The horizontal rows of a matrix are numbered consecutively from top to bottom, and the vertical columns are numbered from left to right. For the foregoing matrix A, we have

$$\begin{array}{r} \\ \text{row 1} \\ \text{row 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{column 1} & \text{column 2} & \text{column 3} \end{array} \\ \begin{bmatrix} 10 & \quad 12 \quad & 16 \\ 5 & 9 & 7 \end{bmatrix} \end{array} = A$$

Since A has two rows and three columns, we say that A *has size* 2×3 (read "2 by 3") or that A *is* 2×3, where the number of rows is specified first. Similarly, the matrices

$$B = \begin{bmatrix} 1 & 6 & -2 \\ 5 & 1 & -4 \\ -3 & 5 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \\ 7 & -8 \end{bmatrix}$$

have sizes 3×3 and 4×2, respectively.

The numbers in a matrix are called its **entries**. To denote the entries in a matrix A of size 2×3, say, we use the name of the matrix, with *double subscripts* to indicate *position*, consistent with the conventions above:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

The row subscript appears to the left of the column subscript. In general, A_{ij} and A_{ji} are different.

For the entry A_{12} (read "A sub one-two" or just "A one-two"), the first subscript, 1, specifies the row and the second subscript, 2, the column in which the entry appears. Similarly, the entry A_{23} (read "A two-three") is the entry in the second row and the third column. Generalizing, we say that the symbol A_{ij} denotes the entry in the ith row and jth column. In fact, a matrix A is a function of two variables with $A(i, j) = A_{ij}$.

If A is $m \times n$, write $\bar{m}$ for the set $\{1, 2, \ldots m\}$. Then, the domain of A is $\bar{m} \times \bar{n}$, the set of all ordered pairs (i, j) with i in $\bar{m}$ and j in $\bar{n}$, while the range is a subset of the set of real numbers, $(-\infty, \infty)$.

Our concern in this chapter is the manipulation and application of various types of matrices. For completeness, we now give a formal definition of a matrix.

Definition

A rectangular array of numbers A consisting of m horizontal rows and n vertical columns,

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

is called an $m \times n$ **matrix** and $m \times n$ is the **size** of A. For the entry A_{ij}, the row subscript is i and the column subscript is j.

The *matrix* $[A_{ij}]$ has A_{ij} as its *general entry*.

The number of entries in an $m \times n$ matrix is mn. For brevity, an $m \times n$ matrix can be denoted by the symbol $[A_{ij}]_{m \times n}$ or, more simply, $[A_{ij}]$, when the size is understood from the context.

A matrix that has exactly one row, such as the 1×4 matrix

$$A = [1 \quad 7 \quad 12 \quad 3]$$

is called a **row vector**. A matrix consisting of a single column, such as the 5×1 matrix

$$\begin{bmatrix} 1 \\ -2 \\ 15 \\ 9 \\ 16 \end{bmatrix}$$

is called a **column vector**. Observe that a matrix is 1×1 if and only if it is both a row vector and a column vector. It is safe to treat 1×1 matrices as mere numbers. In other words, we can write $[7] = 7$, and, more generally, $[a] = a$, for any real number a.

APPLY IT ▶

1. A manufacturer who uses raw materials A and B is interested in tracking the costs of these materials from three different sources. What is the size of the matrix she would use?

EXAMPLE 1 Size of a Matrix

a. The matrix $[1 \quad 2 \quad 0]$ has size 1×3.

b. The matrix $\begin{bmatrix} 1 & -6 \\ 5 & 1 \\ 9 & 4 \end{bmatrix}$ has size 3×2.

c. The matrix $[7]$ has size 1×1.

d. The matrix $\begin{bmatrix} 1 & 3 & 7 & -2 & 4 \\ 9 & 11 & 5 & 6 & 8 \\ 6 & -2 & -1 & 1 & 1 \end{bmatrix}$ has size 3×5 and $(3)(5) = 15$ entries.

Now Work Problem 1a ◁

APPLY IT ▶

2. An analysis of a workplace uses a 3×5 matrix to describe the time spent on each of three phases of five different projects. Project 1 requires 1 hour for each phase, project 2 requires twice as much time as project 1, project 3 requires twice as much time as project 2, . . . , and so on. Construct this time-analysis matrix.

EXAMPLE 2 **Constructing Matrices**

a. Construct a three-entry column matrix A such that $A_{21} = 6$ and $A_{i1} = 0$ otherwise.

Solution: Since $A_{11} = A_{31} = 0$, the matrix is

$$A = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

b. If $[A_{ij}]$ is 3×4 and $A_{ij} = i + j$, find A.

Solution: Here $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, and A has $(3)(4) = 12$ entries. Since $A_{ij} = i + j$, the entry in row i and column j is obtained by adding the numbers i and j. Hence, $A_{11} = 1 + 1 = 2$, $A_{12} = 1 + 2 = 3$, $A_{13} = 1 + 3 = 4$, and so on. Thus,

$$A = \begin{bmatrix} 1+1 & 1+2 & 1+3 & 1+4 \\ 2+1 & 2+2 & 2+3 & 2+4 \\ 3+1 & 3+2 & 3+3 & 3+4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

c. Construct the 3×3 matrix I, given that $I_{11} = I_{22} = I_{33} = 1$ and $I_{ij} = 0$ otherwise.

Solution: The matrix is given by

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now Work Problem 11 ◁

Equality of Matrices

We now define what is meant by saying that two matrices are *equal*.

Definition

Matrices A and B are **equal** if and only if they have the same size and $A_{ij} = B_{ij}$ for each i and j (that is, corresponding entries are equal).

Thus,

$$\begin{bmatrix} 1+1 & \frac{2}{2} \\ 2 \cdot 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 0 \end{bmatrix}$$

but

$$[1 \quad 1] \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad [1 \quad 1] \neq [1 \quad 1 \quad 1] \qquad \text{different sizes}$$

A matrix equation can define a system of equations. For example, suppose that

$$\begin{bmatrix} x & y+1 \\ 2z & 5w \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 4 & 2 \end{bmatrix}$$

By equating corresponding entries, we must have

$$\begin{cases} x = 2 \\ y + 1 = 7 \\ 2z = 4 \\ 5w = 2 \end{cases}$$

Solving gives $x = 2$, $y = 6$, $z = 2$, and $w = \frac{2}{5}$.

Transpose of a Matrix

If A is a matrix, the matrix formed from A by interchanging its rows with its columns is called the *transpose* of A.

> **Definition**
>
> The **transpose** of an $m \times n$ matrix A, denoted A^T, is the $n \times m$ matrix whose ith row is the ith column of A.

EXAMPLE 3 Transpose of a Matrix

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, find A^T.

Solution: Matrix A is 2×3, so A^T is 3×2. Column 1 of A becomes row 1 of A^T, column 2 becomes row 2, and column 3 becomes row 3. Thus,

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Now Work Problem 19 ◁

Observe that the columns of A^T are the rows of A. Also, if we take the transpose of our answer, the original matrix A is obtained. That is, the transpose operation has the property that

$$(A^T)^T = A$$

Special Matrices

Certain types of matrices play important roles in matrix theory. We now consider some of these special types.

An $m \times n$ matrix whose entries are all 0 is called the $m \times n$ **zero matrix** and is denoted by $0_{m\times n}$ or, more simply, by 0 if its size is understood. Thus, the 2×3 zero matrix is

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and, in general, we have

$$0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

A matrix having the same number of columns as rows—for example, n rows and n columns—is called a **square matrix** of order n. That is, an $m \times n$ matrix is square if and only if $m = n$. For example, matrices

$$\begin{bmatrix} 2 & 7 & 4 \\ 6 & 2 & 0 \\ 4 & 6 & 1 \end{bmatrix} \quad \text{and} \quad [3]$$

are square with orders 3 and 1, respectively.