Essential Mathematics for Economic Analysis

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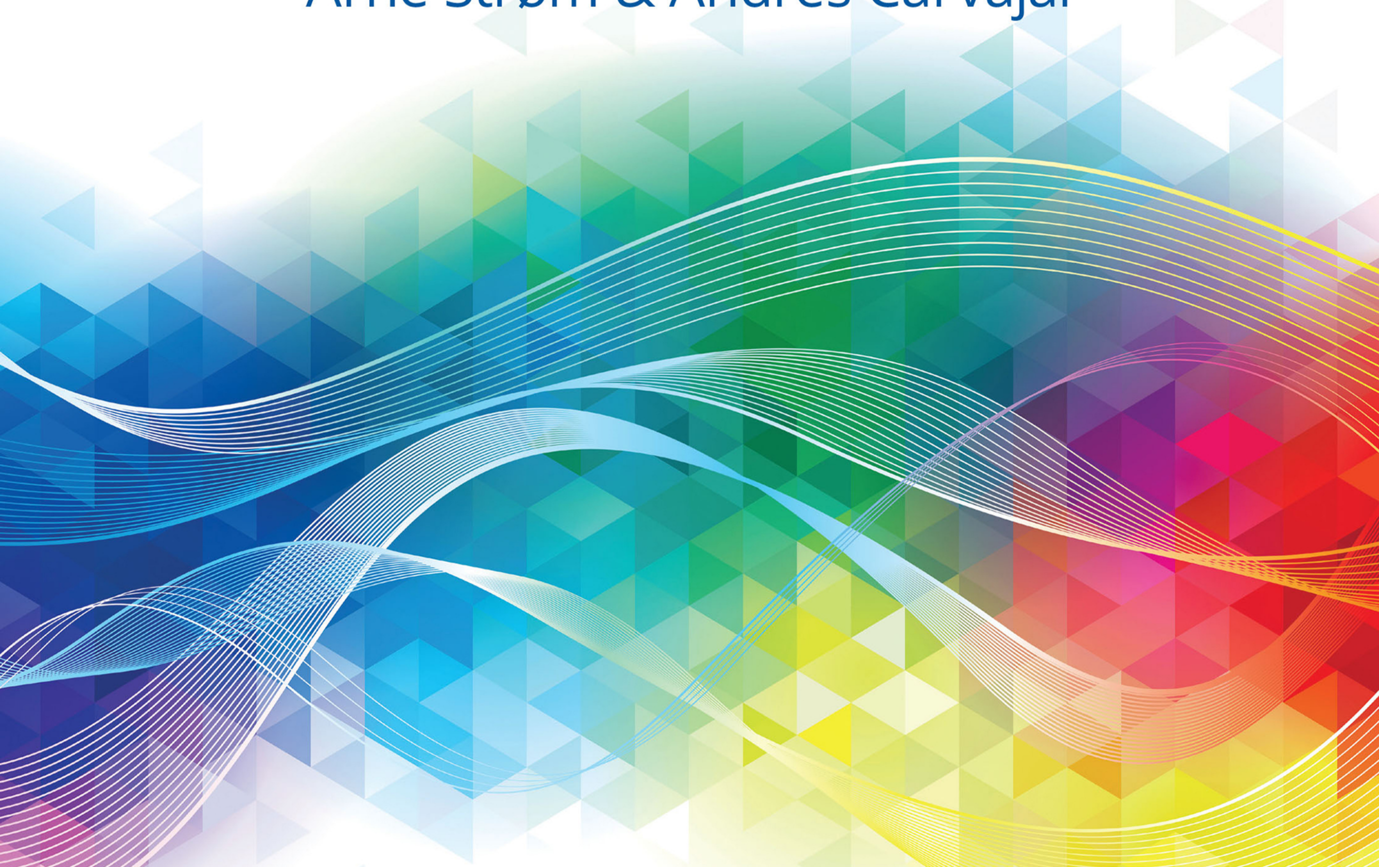

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ESSENTIAL MATHEMATICS FOR

ECONOMIC ANALYSIS

is *left continuous* at a. Similarly, if f is defined on $[a, d)$, we say that f is *right continuous* at a if $f(x)$ tends to $f(a)$ as x tends to a^+. Because of (7.9.1), we see that a function f is continuous at a if and only if f is both left and right continuous at a.

EXAMPLE 7.9.4 Consider again the function f whose graph is shown in Fig. 7.9.3. Because $\lim_{x\to 4^+} f(x)$ exists and is equal to $f(4) = 3$, it follows that f is right continuous at $x = 4$. But at $x = 2$ one has $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3$, yet a dot in the graph indicates that $f(2) = 2$. It follows that f is neither right nor left continuous at $x = 2$.

Consider a function f which is defined on a closed bounded interval $[a, b]$. We usually say that f is continuous in $[a, b]$ if it is not only continuous at each point of the open interval (a, b), but also both right continuous at a and left continuous at b. It should be obvious how to define continuity on half-open intervals. The continuity of a function at all points of an interval (including one-sided continuity at the end points) is often a minimum requirement we impose when speaking about "well-behaved" functions.

Limits at Infinity

We can also use the language of limits to describe the behaviour of a function as its argument becomes infinitely large through positive or negative values. Let f be defined for arbitrarily large positive numbers x. We say that $f(x)$ *has the limit A as x tends to infinity* if $f(x)$ can be made arbitrarily close to A by making x sufficiently large. We write

$$\lim_{x\to\infty} f(x) = A \quad \text{or} \quad f(x) \to A \ \text{ as } \ x \to \infty$$

In the same way,

$$\lim_{x\to -\infty} f(x) = B \quad \text{or} \quad f(x) \to B \ \text{ as } \ x \to -\infty$$

indicates that $f(x)$ can be made arbitrarily close to B by making x a sufficiently large negative number. The two limits are illustrated in Fig. 7.9.4. The horizontal line $y = A$ is a *horizontal asymptote* for the graph of f as x tends to ∞, whereas $y = B$ is a horizontal asymptote for the graph as x tends to $-\infty$.

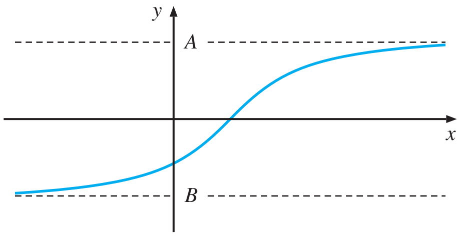

Figure 7.9.4 $y = A$ and $y = B$ are horizontal asymptotes

We remark that the limit $\lim_{x\to -\infty} e^x = 0$ has already been discussed when justifying one of the two limits appearing in Eq. (6.10.3). We also note that $\lim_{x\to +\infty} e^x = +\infty$ is an infinite limit that appeared in Eq. (6.10.3), whereas $\lim_{x\to+\infty} \ln x = +\infty$ appeared in Eq. (6.11.4).

EXAMPLE 7.9.5 Examine the following functions as $x \to \infty$ and as $x \to -\infty$:

(a) $f(x) = \dfrac{3x^2 + x - 1}{x^2 + 1}$ (b) $g(x) = \dfrac{1 - x^5}{x^4 + x + 1}$

Solution:

(a) Away from $x = 0$ we can divide each term in the numerator and the denominator by the highest power of x, which is x^2, to obtain

$$f(x) = \frac{3x^2 + x - 1}{x^2 + 1} = \frac{3 + (1/x) - (1/x^2)}{1 + (1/x^2)}$$

If x is large in absolute value, then both $1/x$ and $1/x^2$ are close to 0. So $f(x)$ is arbitrarily close to 3 if $|x|$ is sufficiently large. It follows that $f(x) \to 3$ both as $x \to -\infty$ and $x \to \infty$.

(b) Note that

$$g(x) = \frac{1 - x^5}{x^4 + x + 1} = \frac{(1/x^4) - x}{1 + (1/x^3) + (1/x^4)}$$

Now you should be able to finish the argument yourself, along the lines given in part (a). One has $g(x) \to +\infty$ as $x \to -\infty$, but $g(x) \to -\infty$ as $x \to +\infty$.

Warnings

We have extended the original definition of a limit in several different directions. For these extended limit concepts, the previous limit rules set out in Section 6.5 still apply. For example, all the usual limit rules are valid if we consider left-hand limits or right-hand limits. Also, if we replace $x \to a$ by $x \to \infty$ or $x \to -\infty$, then again the corresponding limit properties hold. Provided at least one of the two limits A and B is nonzero, the four rules in (6.5.2)–(6.5.5) remain valid if at most one of A and B is infinite.

When $f(x)$ and $g(x)$ both tend to ∞ as x tends to a, however, much more care is needed. Because $f(x)$ and $g(x)$ can each be made arbitrarily large if x is sufficiently close to a, both $f(x) + g(x)$ and $f(x)g(x)$ can also be made arbitrarily large. But, in general, we cannot say what are the limits of $f(x) - g(x)$ and $f(x)/g(x)$. The limits of these expressions will depend on how "fast" $f(x)$ and $g(x)$, respectively, tend to ∞ as x tends to a. Briefly formulated:

PROPERTIES OF INFINITE LIMITS

If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then

$$f(x) + g(x) \to \infty \quad \text{and} \quad f(x)g(x) \to \infty \quad \text{as} \quad x \to a \tag{7.9.2}$$

However, there is no rule for the limits of $f(x) - g(x)$ and $f(x)/g(x)$ as $x \to a$.

Thus, it is important to note that the limits of $f(x) - g(x)$ and $f(x)/g(x)$ cannot be determined without more information about f and g. We do not even know whether or not either of these limits exists. The following example illustrates some of the possibilities.

EXAMPLE 7.9.6 Let $f(x) = 1/x^2$ and $g(x) = 1/x^4$. As $x \to 0$, so $f(x) \to \infty$ and $g(x) \to \infty$. Examine the limit as $x \to 0$ for each of the following four expressions:

$$f(x) - g(x), \quad g(x) - f(x), \quad \frac{f(x)}{g(x)}, \quad \text{and} \quad \frac{g(x)}{f(x)}$$

Solution: $f(x) - g(x) = (x^2 - 1)/x^4$. As $x \to 0$, the numerator tends to -1 and the denominator is positive and tends to 0, so the fraction tends to $-\infty$. For the other three limits as $x \to 0$, we have:

$$g(x) - f(x) = \frac{1 - x^2}{x^4} \to \infty, \quad \frac{f(x)}{g(x)} = x^2 \to 0, \quad \text{and} \quad \frac{g(x)}{f(x)} = \frac{1}{x^2} \to \infty$$

The above four examples serve to illustrate that infinite limits require extreme care. Other tricky examples involve the product $f(x)g(x)$ of two functions, where $g(x)$ tends to 0 as x tends to a. Will the product $f(x)g(x)$ also tend to 0? Not necessarily. If $f(x)$ tends to a finite limit A, then we know that $f(x)g(x)$ tends to $A \cdot 0 = 0$. But if $f(x)$ tends to $\pm\infty$, then it is easy to construct examples in which the product $f(x)g(x)$ does not tend to 0 at all. You should try to construct some examples of your own before turning to Exercise 4.

Continuity and Differentiability

Consider the function f whose graph appears in Fig. 7.9.5. At the point $(a, f(a))$ the graph does not have a unique tangent. Thus f has no derivative at $x = a$, even though f is continuous at $x = a$. So a function can be continuous at a point without being differentiable at that point. A standard example is the absolute value function whose graph is shown in Fig. 4.3.10: that function is continuous everywhere, but not differentiable at the origin.

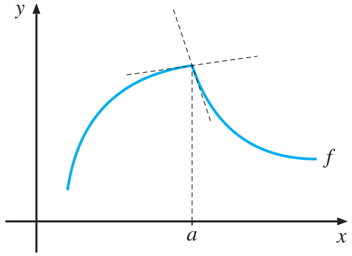

Figure 7.9.5 f is continuous, but not differentiable at $x = a$

On the other hand, differentiability implies continuity:

CONTINUITY AND DIFFERENTIABILITY

If a function f is differentiable at $x = a$, then it is continuous at $x = a$. (7.9.3)

The proof of this result is, in fact, not difficult:

Proof: The function f is continuous at $x = a$ provided $f(a+h) - f(a)$ tends to 0 as $h \to 0$. Now, for all $h \neq 0$, it is trivial that

$$f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \cdot h \qquad (*)$$

If f is differentiable at $x = a$, then by definition the Newton quotient $[f(a+h) - f(a)]/h$ tends to the number $f'(a)$ as $h \to 0$. So the right-hand side of $(*)$ tends to $f'(a) \cdot 0 = 0$ as $h \to 0$. This proves that f is continuous at $x = a$. ▮

Suppose that f is some function whose Newton quotient tends to a limit as h tends to 0 through positive values. Then the limit is called the *right derivative* of f at a. The *left derivative* of f at a is defined similarly. So when the relevant one-sided limits of the Newton quotient exist, we denote them by

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \qquad (7.9.4)$$

Suppose that the function f is continuous at a and has left and right derivatives that satisfy $f'(a^+) \neq f'(a^-)$. In this case when the two derivatives differ, the graph of f is said to have a *kink* at $(a, f(a))$.

EXAMPLE 7.9.7 **(US Federal Income Tax, 2018).** Let $\tau(x)$ denotes the tax liability of somebody whose income during 2018 was x, both amounts measured in US dollars. This income tax function τ was discussed in Example 5.4.4, and its graph illustrated in Fig. 5.4.9. This graph has kinks at, for instance, both $x = 13\,600$ and $x = 51\,800$. Indeed, the tax rate on incomes below \$13 600 was 10%, whereas a taxpayer with an income between \$13 600 and \$51 800 paid 10% of the "first" \$13 600 plus 12% of any income above \$13 600. Thus, there is a kink at \$13 600 with $\tau'(13\,600^-) = 0.1$ and $\tau'(13\,600^+) = 0.12$. Similarly, another kink occurs at \$51 800 where $\tau'(51\,800^-) = 0.12$ and $\tau'(51\,800^+) = 0.22$. The highest kink was at \$500 000 where the highest marginal rate of tax kicks in. Because this is 37%, one has $\tau'(500\,000^+) = 0.37$. ▮

A Rigorous Definition of Limits

In our preliminary definition (6.5.1) of the limit concept, we interpreted $\lim_{x \to a} f(x) = A$ to mean that we can make $f(x)$ as close to A as we want by choosing x sufficiently close (but not equal) to a. Now we make the notion of closeness more precise, following Eq. (2.7.2): two numbers y and z are close if the distance $|y - z|$ between them is small. This allows our preliminary definition to be reformulated as follows:

LIMIT

> $\lim_{x \to a} f(x) = A$ means that we can make $|f(x) - A|$ as small as we want for all $x \neq a$ with $|x - a|$ sufficiently small.

Towards the end of the 19th century some of the world's best mathematicians gradually realized that this definition can be made precise in the following way:[14]

THE $\varepsilon - \delta$ DEFINITION OF LIMIT

We say that $f(x)$ has limit A as x tends to a if, for each number $\varepsilon > 0$, there exists an associated number $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x with $0 < |x - a| < \delta$. When this holds, we say that $f(x)$ tends to A as x tends to a, and write

$$\lim_{x \to a} f(x) = A$$

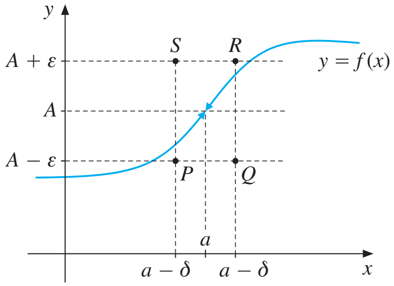

Figure 7.9.6 Definition of limit

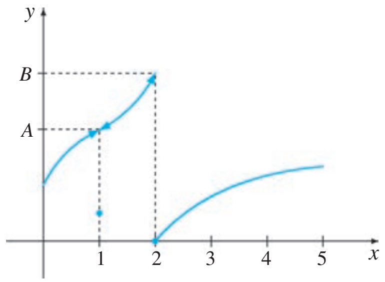

Figure 7.9.7 Exercise 1

This definition forms the basis of all mathematically rigorous work on limits. Figure 7.9.6 illustrates the definition. In the figure it implies that, for every $\varepsilon > 0$ and every associated δ, the graph of f must remain within the rectangular box $PQRS$, for all $x \neq a$ in $(a - \delta, a + \delta)$. In particular, the graph cannot pass from the interior of the box to its exterior by crossing either of the horizontal line segments PQ and SR; instead, it must cross the vertical line segments PS and QR.

Seeing this formal ε–δ definition of a limit should be regarded as a part of anybody's general mathematical education. In this book, however, we rely only on an intuitive understanding of limits.

EXERCISES FOR SECTION 7.9

1. Function f, defined for $0 < x < 5$, has the graph that appears in Fig. 7.9.7. Determine the following limits:

(a) $\lim_{x \to 1^-} f(x)$ (b) $\lim_{x \to 1^+} f(x)$ (c) $\lim_{x \to 2^-} f(x)$ (d) $\lim_{x \to 2^+} f(x)$

[14] This specific idea is often attributed to the two German mathematicians Eduard Heine (1821–1881) and Karl Weierstrass (1815–1897), although really there is no consensus about this.

SM **2.** Evaluate the following limits:

(a) $\lim_{x\to 0^+}(x^2+3x-4)$ (b) $\lim_{x\to 0^-}\frac{x+|x|}{x}$ (c) $\lim_{x\to 0^+}\frac{x+|x|}{x}$

(d) $\lim_{x\to 0^+}\frac{-1}{\sqrt{x}}$ (e) $\lim_{x\to 3^+}\frac{x}{x-3}$ (f) $\lim_{x\to 3^-}\frac{x}{x-3}$

3. Evaluate

(a) $\lim_{x\to\infty}\frac{x-3}{x^2+1}$ (b) $\lim_{x\to -\infty}\sqrt{\frac{2+3x}{x-1}}$ (c) $\lim_{x\to\infty}\frac{(ax-b)^2}{(a-x)(b-x)}$

4. Let $f_1(x)=x$, $f_2(x)=x$, $f_3(x)=x^2$, and $f_4(x)=1/x$. Determine $\lim_{x\to\infty}f_i(x)$ for $i=1,2,3,4$. Then examine whether the rules for limits in Section 6.5 apply to the following limits as $x\to\infty$.

(a) $f_1(x)+f_2(x)$ (b) $f_1(x)-f_2(x)$ (c) $f_1(x)-f_3(x)$ (d) $f_1(x)/f_2(x)$

(e) $f_1(x)/f_3(x)$ (f) $f_1(x)f_2(x)$ (g) $f_1(x)f_4(x)$ (h) $f_3(x)f_4(x)$

SM **5.** The line $y=ax+b$ is said to be an *asymptote* as $x\to\infty$ (or $x\to-\infty$) to the curve $y=f(x)$ if

$$f(x)-(ax+b)\to 0 \text{ as } x\to\infty \quad (\text{or } x\to-\infty)$$

This condition means that the vertical distance between any point $(x,f(x))$ on the curve and the corresponding point $(x,ax+b)$ on the line tends to 0 as $x\to\pm\infty$, as shown in Fig. 7.9.8.

Suppose that $f(x)=P(x)/Q(x)$ is a rational function where the degree of the polynomial $P(x)$ is exactly one greater than that of the polynomial $Q(x)$. In this case $f(x)$ will have an asymptote. Indeed, to find this one begins by performing the polynomial division $P(x)\div Q(x)$ with remainder that was explained in Section 4.7. The result will be a polynomial of degree 1, plus a remainder term that tends to 0 as $x\to\pm\infty$. Use this method to find asymptotes for the graph of each of the following functions of x:

(a) $\frac{x^2}{x+1}$ (b) $\frac{2x^3-3x^2+3x-6}{x^2+1}$ (c) $\frac{3x^2+2x}{x-1}$ (d) $\frac{5x^4-3x^2+1}{x^3-1}$

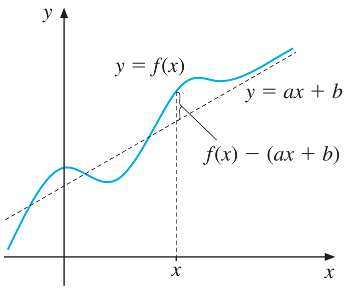

Figure 7.9.8 Exercise 5

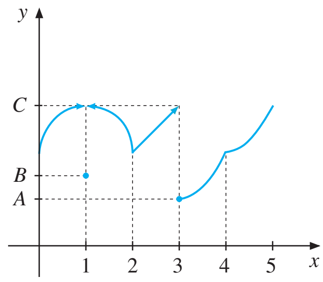

Figure 7.9.9 Exercise 7

6. Consider the cost function defined for all $x\geq 0$ by

$$C(x)=A\frac{x(x+b)}{x+c}+d$$

where A, b, c, and d are positive constants. Find its asymptotes.