GLOBAL EDITION

AN INTRODUCTION TO ANALYSIS

FOURTH EDITION

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An Introduction to Analysis

By (3) it is clear that $\{s_n\}$ is bounded if and only if $\{t_n\}$ is. Since $f(x) \geq 0$ implies that both s_n and t_n are increasing sequences, it follows from the Monotone Convergence Theorem that s_n converges if and only if t_n converges, as $n \to \infty$. ■

This test works best on series for which the integral of f can be easily computed or estimated. For example, to find out whether $\sum_{k=1}^{\infty} 1/(1+k^2)$ converges or diverges, let $f(x) = 1/(1+x^2)$ and observe that f is positive on $[1, \infty)$. Since $f'(x) = -2x/(1+x^2)^2$ is negative on $[1, \infty)$, it is also clear that f is decreasing. Since

$$\int_1^{\infty} \frac{dx}{1+x^2} = \arctan x \;\Big|_1^{\infty} = \frac{\pi}{2} - \arctan(1) < \infty,$$

it follows from the Integral Test that $\sum_{k=1}^{\infty} 1/(1+k^2)$ converges.

The Integral Test is most widely used in the following special case.

6.13 Corollary. [*p*- SERIES TEST]. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \tag{4}$$

converges if and only if $p > 1$.

Proof. If $p = 1$ or $p \leq 0$, the series diverges. If $p > 0$ and $p \neq 1$, set $f(x) = x^{-p}$ and observe that $f'(x) = -px^{-p-1} < 0$ for all $x \in [1, \infty)$. Hence, f is nonnegative and decreasing on $[1, \infty)$. Since

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{n\to\infty} \frac{x^{1-p}}{1-p} \;\Big|_1^{n} = \lim_{n\to\infty} \frac{n^{1-p} - 1}{1-p}$$

has a finite limit if and only if $1 - p < 0$, it follows from the Integral Test that (4) converges if and only if $p > 1$. ■

The Integral Test, which requires f to satisfy some very restrictive hypotheses, has limited applications. The following test can be used in a much broader context.

6.14 Theorem. [COMPARISON TEST].
Suppose that $0 \leq a_k \leq b_k$ *for large* k.

i) *If* $\sum_{k=1}^{\infty} b_k < \infty$, *then* $\sum_{k=1}^{\infty} a_k < \infty$.

ii) *If* $\sum_{k=1}^{\infty} a_k = \infty$, *then* $\sum_{k=1}^{\infty} b_k = \infty$.

Proof. By hypothesis, choose $N \in \mathbf{N}$ so large that $0 \le a_k \le b_k$ for $k > N$. Set $s_n = \sum_{k=1}^n a_k$ and $t_n = \sum_{k=1}^n b_k$, $n \in \mathbf{N}$. Then $0 \le s_n - s_N \le t_n - t_N$ for all $n \ge N$. Since N is fixed, it follows that s_n is bounded when t_n is, and t_n is unbounded when s_n is. Apply Theorem 6.11 and the proof of the theorem is complete. ■

The Comparison Test is used to compare one series with another whose convergence property is already known (e.g., a p-series or a geometric series). Frequently, the inequalities $|\sin x| \le |x|$ for all $x \in \mathbf{R}$ (see Appendix B) and $|\log x| \le x^\alpha$ for each $\alpha > 0$ provided x is sufficiently large (see Exercise 4.4.6) are helpful in this regard. Although there is no simple algorithm for this process, the idea is to examine the terms of the given series, ignoring the superfluous factors, and dominating the more complicated factors by simpler ones. Here is a typical example.

6.15 *EXAMPLE.*

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{3k}{k^2+k}\sqrt{\frac{\log k}{k}} \tag{5}$$

converges or diverges.

Solution. The kth term of this series can be written by using three factors:

$$\frac{1}{k}\frac{3k}{k+1}\sqrt{\frac{\log k}{k}}.$$

The factor $3k/(k+1)$ is dominated by 3. Since $\log k \le \sqrt{k}$ for large k, the factor $\sqrt{\log k/k}$ satisfies

$$\sqrt{\frac{\log k}{k}} \le \sqrt{\frac{\sqrt{k}}{k}} = \frac{1}{\sqrt[4]{k}}$$

for large k. Therefore, the terms of (5) are dominated by $3/k^{5/4}$. Since $\sum_{k=1}^\infty 3/k^{5/4}$ converges by the p-Series Test, it follows from the Comparison Test that (5) converges. ■

The Comparison Test may not be easy to apply to a given series, even when we know which series it should be compared with, because the process of comparison often involves use of delicate inequalities. For situations like this, the following test is usually more efficient.

6.16 Theorem. [LIMIT COMPARISON TEST].
Suppose that $a_k \geq 0$, that $b_k > 0$ for large k, and that $L := \lim_{n\to\infty} a_n/b_n$ exists as an extended real number.

i) *If $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.*

ii) *If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*

iii) *If $L = \infty$ and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. i) If L is finite and nonzero, then there is an $N \in \mathbf{N}$ such that

$$\frac{L}{2} b_k < a_k < \frac{3L}{2} b_k$$

for $k \geq N$. Hence, part i) follows immediately from the Comparison Test and Theorem 6.10. Similar arguments establish parts ii) and iii)—see Exercise 6.2.6. ∎

In general, the Limit Comparison Test is used to replace a series $\sum_{k=1}^{\infty} a_k$ by $\sum_{k=1}^{\infty} b_k$ when $a_k \approx C b_k$ for k large and some absolute fixed constant C. For example, to determine whether or not the series

$$S := \sum_{k=1}^{\infty} \frac{k}{\sqrt{4k^4 + k^2} + 5k}$$

converges, notice that its terms are approximately $1/(2k)$ for k large. This leads us to compare S with the harmonic series $\sum_{k=1}^{\infty} 1/k$. Since the harmonic series diverges and since

$$\frac{k/(\sqrt{4k^4 + k^2} + 5k)}{1/k} = \frac{k^2}{\sqrt{4k^4 + k^2} + 5k} \to \frac{1}{2} > 0$$

as $k \to \infty$, it follows from the Limit Comparison Test that S diverges.

Here is another application of the Limit Comparison Test.

6.17 *EXAMPLE.*

Let $a_k \to 0$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty} \sin|a_k|$ converges if and only if $\sum_{k=1}^{\infty} |a_k|$ converges.

Proof. By l'Hôpital's Rule,

$$\lim_{k\to\infty} \frac{\sin|a_k|}{|a_k|} = \lim_{x\to 0+} \frac{\sin x}{x} = 1.$$

Hence, by the Limit Comparison Test, $\sum_{k=1}^{\infty} \sin|a_k|$ converges if and only if $\sum_{k=1}^{\infty} |a_k|$ converges. ∎

EXERCISES

6.2.0. Let $\{a_k\}$ and $\{b_k\}$ be real sequences. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples to the false ones.

a) If $\sum_{k=1}^{\infty} a_k$ converges and $a_k/b_k \to 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty} b_k$ converges.

b) Suppose that $0 < a < 1$. If $a_k \geq 0$ and $\sqrt[k]{a_k} \leq a$ for all $k \in \mathbf{N}$, then $\sum_{k=1}^{\infty} a_k$ converges.

c) Suppose that $a_k \to 0$ as $k \to \infty$. If $a_k \geq 0$ and $\sqrt{a_{k+1}} \leq a_k$ for all $k \in \mathbf{N}$, then $\sum_{k=1}^{\infty} a_k$ converges.

d) Suppose that $a_k = f(k)$ for some continuous function $f : [1, \infty) \to [0, \infty)$ which satisfies $f(x) \to 0$ as $x \to \infty$. If $\sum_{k=1}^{\infty} a_k$ converges, then $\int_1^{\infty} f(x)dx$ converges.

6.2.1. Prove that each of the following series converges.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{5k+2}{2k^3 - 4k + 5}$

b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{e^k}$

c) $\displaystyle\sum_{k=1}^{\infty} \frac{\log k}{k^p}, \; p > 1$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{2k^5 \log^2 k}{8!\, e^{k+1}}$

e) $\displaystyle\sum_{k=1}^{\infty} \frac{e^2 + \sqrt[3]{k}}{\pi + \sqrt[6]{k^5}}$

f) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\log k}}$

6.2.2. Prove that each of the following series diverges.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{3k^3 + 2k^2 + k}{4k^4 - 2k^2 + 1}$

b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[k]{k}}{k}$

c) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{k+2}{k+1}\right)^k$

d) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \log^p k}, \; p \leq 1$

6.2.3. If $a_k \geq 0$ is a bounded sequence, prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{(k+1)^p}$$

converges for all $p > 1$.

6.2.4. Find all $p \geq 0$ such that the following series converges:

$$\sum_{k=1}^{\infty} \frac{1}{k \log^p (k+1)}.$$

6.2.5. If $\sum_{k=1}^{\infty} |a_k|$ converges, prove that

$$\sum_{k=1}^{\infty} \frac{|a_k|}{k^p}$$

converges for all $p \geq 0$. What happens if $p < 0$?

6.2.6. Prove Theorem 6.16ii and iii.

6.2.7. Suppose that a_k and b_k are nonnegative for all $k \in \mathbf{N}$. Prove that if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{\infty} a_k b_k$ also converges.

6.2.8. Suppose that $a, b \in \mathbf{R}$ satisfy $b/a \in \mathbf{R} \backslash \mathbf{Z}$. Find all $q > 0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{(ak+b)q^k}$$

converges.

6.2.9. Suppose that $a_k \to 0$. Prove that $\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\sum_{k=1}^{\infty} (a_{2k} + a_{2k+1})$ converges.

6.2.10. Find all $p \in \mathbf{R}$ such that

$$\sum_{k=2}^{\infty} \frac{1}{(\log(\log k))^{p \log k}}$$

converges.

6.3 ABSOLUTE CONVERGENCE

In this section we investigate what happens to a convergent series when its terms are replaced by their absolute values. We begin with some terminology.

6.18 Definition.

Let $S = \sum_{k=1}^{\infty} a_k$ be an infinite series.

i) S is said to *converge absolutely* if and only if $\sum_{k=1}^{\infty} |a_k| < \infty$.

ii) S is said to *converge conditionally* if and only if S converges but not absolutely.

The Cauchy Criterion gives us the following test for absolute convergence.

6.19 Remark. *A series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely if and only if for every* $\varepsilon > 0$ *there is an* $N \in \mathbf{N}$ *such that*

$$m > n \geq N \quad \text{implies} \quad \sum_{k=n}^{m} |a_k| < \varepsilon. \tag{6}$$

As was the case for improper integrals, absolute convergence is stronger than convergence.

6.20 Remark. *If* $\sum_{k=1}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} a_k$ *converges, but not conversely. In particular, there exist conditionally convergent series.*

Proof. Suppose that $\sum_{k=1}^{\infty} a_k$ converges absolutely. Given $\varepsilon > 0$, choose $N \in \mathbf{N}$ so that (6) holds. Then

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \varepsilon$$

for $m > n \geq N$. Hence, by the Cauchy Criterion, $\sum_{k=1}^{\infty} a_k$ converges.

We shall finish the proof by showing that $S := \sum_{k=1}^{\infty} (-1)^k/k$ converges conditionally. Since the harmonic series diverges, S does not converge absolutely. On the other hand, the tails of S look like

$$\sum_{j=k}^{\infty} \frac{(-1)^j}{j} = (-1)^k \left(\frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+2} - \frac{1}{k+3} + \dots \right).$$

By grouping pairs of terms together, it is easy to see that the sum inside the parentheses is greater than 0 but less than $1/k$; that is,

$$\left| \sum_{j=k}^{\infty} \frac{(-1)^j}{j} \right| < \frac{1}{k}.$$

Hence $\sum_{k=1}^{\infty} (-1)^k/k$ converges by Corollary 6.9. ∎

We shall see below that it is important to be able to identify absolutely convergent series. Since every result about series with nonnegative terms can be applied to the series $\sum_{k=1}^{\infty} |a_k|$, we already have three tests for absolute convergence (the Integral Test, the Comparison Test, and the Limit Comparison Test). We now develop two additional tests for absolute convergence which are arguably the most practical tests presented in this chapter.

Before we state these tests, we need to introduce another concept. (If you covered Section 2.5, you may proceed directly to Theorem 6.23.)