GLOBAL EDITION

Differential Equations & Linear Algebra

FOURTH EDITION

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DIFFERENTIAL EQUATIONS & LINEAR ALGEBRA

Fourth Edition
Global Edition

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determines the vector $\mathbf{v} = (a, b, c)$ in $\mathbf{R}^3$, and $\mathbf{v}$ is represented geometrically (as in Fig. 4.1.1) by the *position vector* $\overrightarrow{OP}$ from the origin $O(0, 0, 0)$ to P—or equally well by any parallel translate of this arrow. What is important about an arrow usually is not where it is, but how long it is and which way it points.

As in Section 3.4, we adopt the convention that the vector $\mathbf{v}$ with components v_1, v_2, and v_3 may be written interchangeably as either

$$\mathbf{v} = (v_1, v_2, v_3) \quad \text{or} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

with the column matrix regarded as just another symbol representing one and the same ordered triple of real numbers. Then the following definitions of addition of vectors and of multiplication of vectors by scalars are consistent with the matrix operations defined in Section 3.4.

DEFINITION Addition of Vectors

The **sum** $\mathbf{u} + \mathbf{v}$ of the two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is the vector

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \tag{1}$$

that is obtained upon addition of respective components of $\mathbf{u}$ and $\mathbf{v}$.

Thus we add vectors by adding corresponding components—that is, by *componentwise addition* (just as we add matrices). For instance, the sum of the vectors $\mathbf{u} = (4, 3, -5)$ and $\mathbf{v} = (-5, 2, 15)$ is the vector

$$\mathbf{u} + \mathbf{v} = (4, 3, -5) + (-5, 2, 15) = (4 - 5, 3 + 2, -5 + 15) = (-1, 5, 10).$$

The geometric representation of vectors as arrows often converts an algebraic relation into a picture that is readily understood and remembered. Addition of vectors is defined algebraically by Eq. (1). The geometric interpretation of vector addition is the *triangle law of addition* illustrated in Fig. 4.1.2 (for the case of 2-dimensional vectors in the plane), where the labeled lengths indicate why this interpretation is valid. An equivalent interpretation is the *parallelogram law of addition*, illustrated in Fig. 4.1.3.

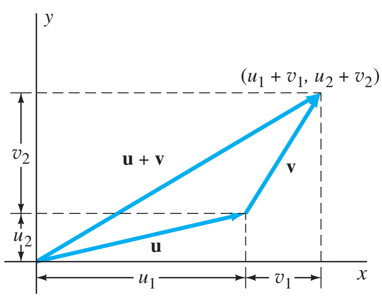

FIGURE 4.1.2. The triangle law of vector addition.

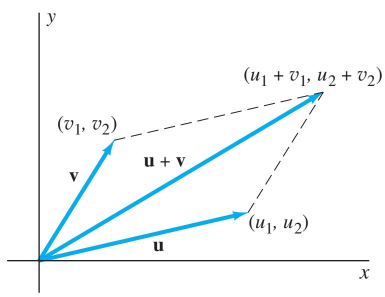

FIGURE 4.1.3. The parallelogram law of vector addition.

Multiplication of a vector by a scalar (a real number) is also defined in a componentwise manner.

DEFINITION Multiplication of a Vector by a Scalar

If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector and c is a real number, then the **scalar multiple** $c\mathbf{v}$ is the vector

$$c\mathbf{v} = (cv_1, cv_2, cv_3) \tag{2}$$

that is obtained upon multiplying each component of $\mathbf{v}$ by c.

The **length** $|\mathbf{v}|$ of the vector $\mathbf{v} = (a, b, c)$ is defined to be the distance of the point $P(a, b, c)$ from the origin,

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}. \tag{3}$$

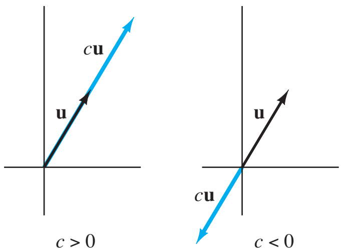

FIGURE 4.1.4. The vector $c\mathbf{u}$ may have the same direction as $\mathbf{u}$ or the opposite direction.

The length of $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$. For instance, if $\mathbf{v} = (4, 3, -12)$, then

$$(-7)\mathbf{v} = (-7 \cdot 4, -7 \cdot 3, -7 \cdot (-12)) = (-28, -21, 84),$$
$$|\mathbf{v}| = \sqrt{(4)^2 + (3)^2 + (-12)^2} = \sqrt{169} = 13, \quad \text{and}$$
$$|-7\mathbf{v}| = |-7| \cdot |\mathbf{v}| = 7 \cdot 13 = 91.$$

The geometric interpretation of scalar multiplication is that $c\mathbf{v}$ is a vector of length $|c| \cdot |\mathbf{v}|$, with the same direction as $\mathbf{v}$ if $c > 0$ but the opposite direction if $c < 0$ (Fig. 4.1.4).

With vector addition and multiplication by scalars defined as in (1) and (2), $\mathbf{R}^3$ is a **vector space**. That is, these operations satisfy the conditions in (a)–(h) of the following theorem.

THEOREM 1 $\mathbf{R}^3$ as a Vector Space

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbf{R}^3$, and r and s are real numbers, then

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associativity)

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ (zero element)

(d) $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$ (additive inverse)

(e) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ (distributivity)

(f) $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$

(g) $r(s\mathbf{u}) = (rs)\mathbf{u}$

(h) $1(\mathbf{u}) = \mathbf{u}$ (multiplicative identity).

Of course, $\mathbf{0} = (0, 0, 0)$ denotes the **zero vector** in (c) and (d), and $-\mathbf{u} = (-1)\mathbf{u}$ in (d). Each of properties (a)–(h) in Theorem 1 is readily verified in a componentwise manner. For instance, if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then use of the (ordinary) distributive law of real numbers gives

$$\begin{aligned} r(\mathbf{u} + \mathbf{v}) &= r(u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ &= (r(u_1 + v_1), r(u_2 + v_2), r(u_3 + v_3)) \\ &= (ru_1 + rv_1, ru_2 + rv_2, ru_3 + rv_3) \\ &= (ru_1, ru_2, ru_3) + (rv_1, rv_2, rv_3) \\ &= r(u_1, u_2, u_3) + r(v_1, v_2, v_3) = r\mathbf{u} + r\mathbf{v}, \end{aligned}$$

so we have verified property (e).

The Vector Space $\mathbf{R}^2$

The familiar coordinate plane $\mathbf{R}^2$ is the set of all ordered pairs (a, b) of real numbers. We may regard $\mathbf{R}^2$ as the xy-plane in $\mathbf{R}^3$ by identifying (a, b) with the vector $(a, b, 0)$ in $\mathbf{R}^3$. Then $\mathbf{R}^2$ is simply the set of all 3-dimensional vectors that have third component 0.

Clearly, the sum of any two vectors in $\mathbf{R}^2$ is again a vector in $\mathbf{R}^2$, as is any scalar multiple of a vector in $\mathbf{R}^2$. Indeed, vectors in $\mathbf{R}^2$ satisfy all the properties of a vector space enumerated in Theorem 1. Consequently, the plane $\mathbf{R}^2$ is a vector space in its own right.

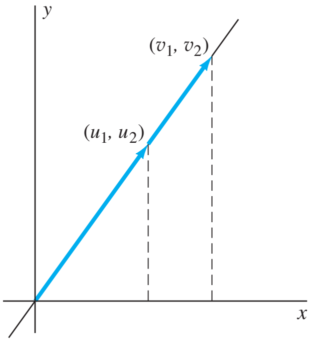

FIGURE 4.1.5. Two linearly dependent vectors **u** and **v**.

The two vectors **u** and **v** are collinear—they lie on the same line through the origin and hence point either in the same direction (Fig. 4.1.5) or in opposite directions—if and only if one is a scalar multiple of the other; that is, either

$$\mathbf{u} = c\mathbf{v} \quad \text{or} \quad \mathbf{v} = c\mathbf{u}, \tag{4}$$

for some scalar c. The scalar c merely adjusts the length and direction of one vector to fit the other. If **u** and **v** are nonzero vectors, then $c = \pm|\mathbf{u}|/|\mathbf{v}|$ in the first relation, with c being positive if the two vectors point in the same direction, negative otherwise.

If one of the relations in (4) holds for some scalar c, then we say that the two vectors are **linearly dependent**. Note that if $\mathbf{u} = \mathbf{0}$ while $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{u} = 0\mathbf{v}$ but **v** is not a scalar multiple of **u**. (Why?) Thus, if precisely one of the two vectors **u** and **v** is the zero vector, then **u** and **v** are linearly dependent, but only one of the two relations in (4) holds.

If **u** and **v** are linearly dependent vectors with $\mathbf{u} = c\mathbf{v}$ (for instance), then $1 \cdot \mathbf{u} + (-c) \cdot \mathbf{v} = \mathbf{0}$. Thus there exist scalars a and b *not both zero* such that

$$a\mathbf{u} + b\mathbf{v} = \mathbf{0}. \tag{5}$$

Conversely, suppose that Eq. (5) holds with a and b not both zero. If $a \neq 0$ (for instance) then we can solve for

$$\mathbf{u} = -\frac{b}{a}\mathbf{v} = c\mathbf{v}$$

with $c = -b/a$, so it follows that **u** and **v** are linearly dependent. Therefore, we have proved the following theorem.

THEOREM 2 Two Linearly Dependent Vectors

The two vectors **u** and **v** are linearly dependent if and only if there exist scalars a and b *not both zero* such that

$$a\mathbf{u} + b\mathbf{v} = \mathbf{0}. \tag{5}$$

The most interesting pairs of vectors are those that are not linearly dependent. The two vectors **u** and **v** are said to be **linearly independent** provided that they are *not* linearly dependent. Thus **u** and **v** are linearly independent if and only if neither is a scalar multiple of the other. By Theorem 2 this is equivalent to the following statement:

> The two vectors **u** and **v** are linearly independent if and only if the relation
>
> $$a\mathbf{u} + b\mathbf{v} = \mathbf{0} \tag{5}$$
>
> implies that $a = b = 0$.

Thus the vectors **u** and **v** are linearly independent provided that no *nontrivial* linear combination of them is equal to the zero vector.

Example 1 If $\mathbf{u} = (3, -2)$, $\mathbf{v} = (-6, 4)$, and $\mathbf{w} = (5, -7)$, then **u** and **v** are linearly dependent, because $\mathbf{v} = -2\mathbf{u}$. On the other hand, **u** and **w** are linearly independent. Here is an argument to establish this fact: Suppose that there were scalars a and b such that

$$a\mathbf{u} + b\mathbf{w} = \mathbf{0}.$$

Then

$$a(3, -2) + b(5, -7) = \mathbf{0},$$

and thus we get the simultaneous equations

$$\begin{aligned} 3a + 5b &= 0 \\ -2a - 7b &= 0. \end{aligned}$$

It is now easy to show that $a = b = 0$ is the (unique) solution of this system. This shows that whenever

$$a\mathbf{u} + b\mathbf{w} = \mathbf{0},$$

it follows that $a = b = 0$. Therefore, **u** and **w** are linearly independent.

Alternatively, we could prove that **u** and **w** are linearly independent by showing that neither is a scalar multiple of the other (because $\frac{5}{3} \neq \frac{7}{2}$). ■

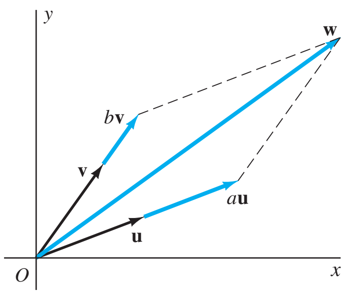

FIGURE 4.1.6. The vector **w** as a linear combination of the two linearly independent vectors **u** and **v**.

The most important property of linearly independent pairs of plane vectors is this: If **u** and **v** are linearly independent vectors in the plane, then any third vector **w** in $\mathbf{R}^2$ can be expressed as a *linear combination* of **u** and **v**. That is, there exist scalars a and b such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ (Fig. 4.1.6). This is a statement that the two linearly independent vectors **u** and **v** suffice (in an obvious sense) to "generate" the whole plane $\mathbf{R}^2$. This general fact—which Section 4.3 discusses in a broader context—is illustrated computationally by the following example.

Example 2 Express the vector $\mathbf{w} = (11, 4)$ as a linear combination of the vectors $\mathbf{u} = (3, -2)$ and $\mathbf{v} = (-2, 7)$.

Solution We want to find numbers a and b such that $a\mathbf{u} + b\mathbf{v} = \mathbf{w}$; that is,

$$a\begin{bmatrix} 3 \\ -2 \end{bmatrix} + b\begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}.$$

This vector equation is equivalent to the 2×2 linear system

$$\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix},$$

which (using Gaussian elimination or Cramer's rule) we readily solve for $a = 5$, $b = 2$. Thus $\mathbf{w} = 5\mathbf{u} + 2\mathbf{v}$. ■

Linear Independence in $\mathbf{R}^3$

We have said that the two vectors are linearly dependent provided that they lie on the same line through the origin. For *three* vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ in space, the analogous condition is that the three points (u_1, u_2, u_3), (v_1, v_2, v_3), and (w_1, w_2, w_3) lie in the same *plane* through the origin in $\mathbf{R}^3$. Given $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, how can we determine whether the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are *coplanar*? The key to the answer is the following observation: If r and s are scalars, then the parallelogram law of addition implies that the vectors $\mathbf{u}$, $\mathbf{v}$, and $r\mathbf{u} + s\mathbf{v}$ are coplanar; specifically, they lie in the plane through the origin that is determined by the parallelogram with vertices $\mathbf{0}$, $r\mathbf{u}$, $s\mathbf{v}$, and $r\mathbf{u} + s\mathbf{v}$. Thus any linear combination of $\mathbf{u}$ and $\mathbf{v}$ is coplanar with $\mathbf{u}$ and $\mathbf{v}$. This is the motivation for our next definition.

DEFINITION Linearly Dependent Vectors in $\mathbf{R}^3$

The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbf{R}^3$ are said to be **linearly dependent** provided that one of them is a linear combination of the other two—that is, either

$$\begin{aligned} \mathbf{w} &= r\mathbf{u} + s\mathbf{v} \qquad \text{or} \\ \mathbf{u} &= r\mathbf{v} + s\mathbf{w} \qquad \text{or} \\ \mathbf{v} &= r\mathbf{u} + s\mathbf{w} \end{aligned} \tag{6}$$

for appropriate scalars r and s.

Note that each of the three equations in (6) implies that there exist three scalars a, b, and c *not all zero* such that

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}. \tag{7}$$

For if $\mathbf{w} = r\mathbf{u} + s\mathbf{v}$ (for instance), then

$$r\mathbf{u} + s\mathbf{v} + (-1)\mathbf{w} = \mathbf{0},$$

so we can take $a = r$, $b = s$, and $c = -1 \neq 0$. Conversely, suppose that (7) holds with a, b, and c not all zero. If $c \neq 0$ (for instance), then we can solve for

$$\mathbf{w} = -\frac{a}{c}\mathbf{u} - \frac{b}{c}\mathbf{v} = r\mathbf{u} + s\mathbf{v}$$

with $r = -a/c$ and $s = -b/c$, so it follows that the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent. Therefore, we have proved the following theorem.

THEOREM 3 Three Linearly Dependent Vectors

The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbf{R}^3$ are linearly dependent if and only if there exist scalars a, b, and c *not all zero* such that

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}. \tag{7}$$

The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are called **linearly independent** provided that they are *not* linearly dependent. Thus $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent if and only if neither of them is a linear combination of the other two. As a consequence of Theorem 3, this is equivalent to the following statement:

> The vectors **u**, **v**, and **w** are linearly independent if and only if the relation
>
> $$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0} \tag{7}$$
>
> implies that $a = b = c = 0$.

Thus the three vectors **u**, **v**, and **w** are linearly independent provided that no *nontrivial* linear combination of them is equal to the zero vector.

Given two vectors, we can see at a glance whether either is a scalar multiple of the other. By contrast, it is not evident at a glance whether or not three given vectors in $\mathbf{R}^3$ are linearly independent. The following theorem provides one way to resolve this question.

THEOREM 4 Three Linearly Independent Vectors

The vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ are linearly independent if and only if

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0. \tag{8}$$

Proof: We want to show that **u**, **v**, and **w** are linearly independent if and only if the matrix

$$\mathbf{A} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

—with column vectors **u**, **v**, and **w**—has nonzero determinant. By Theorem 3, the vectors **u**, **v**, and **w** are linearly independent if and only if the equation

$$a\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + b\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + c\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \mathbf{0}$$

implies that $a = b = c = 0$—that is, if and only if the system

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

with unknowns a, b, and c has only the trivial solution. But Theorem 7 in Section 3.5 implies that this is so if and only if the coefficient matrix **A** is invertible, and Theorem 2 in Section 3.6 implies that **A** is invertible if and only if $|\mathbf{A}| \neq 0$. Therefore, **u**, **v**, and **w** are linearly independent if and only if $|\mathbf{A}| \neq 0$. ▲

Hence, in order to determine whether or not three given vectors **u**, **v**, and **w** are linearly independent, we can calculate the determinant in (8). In practice, however, it is usually more efficient to set up and solve the linear system in (9). If we obtain only the trivial solution $a = b = c = 0$, then the three given vectors are linearly