MODERN ENGINEERING MATHEMATICS

Sixth Edition

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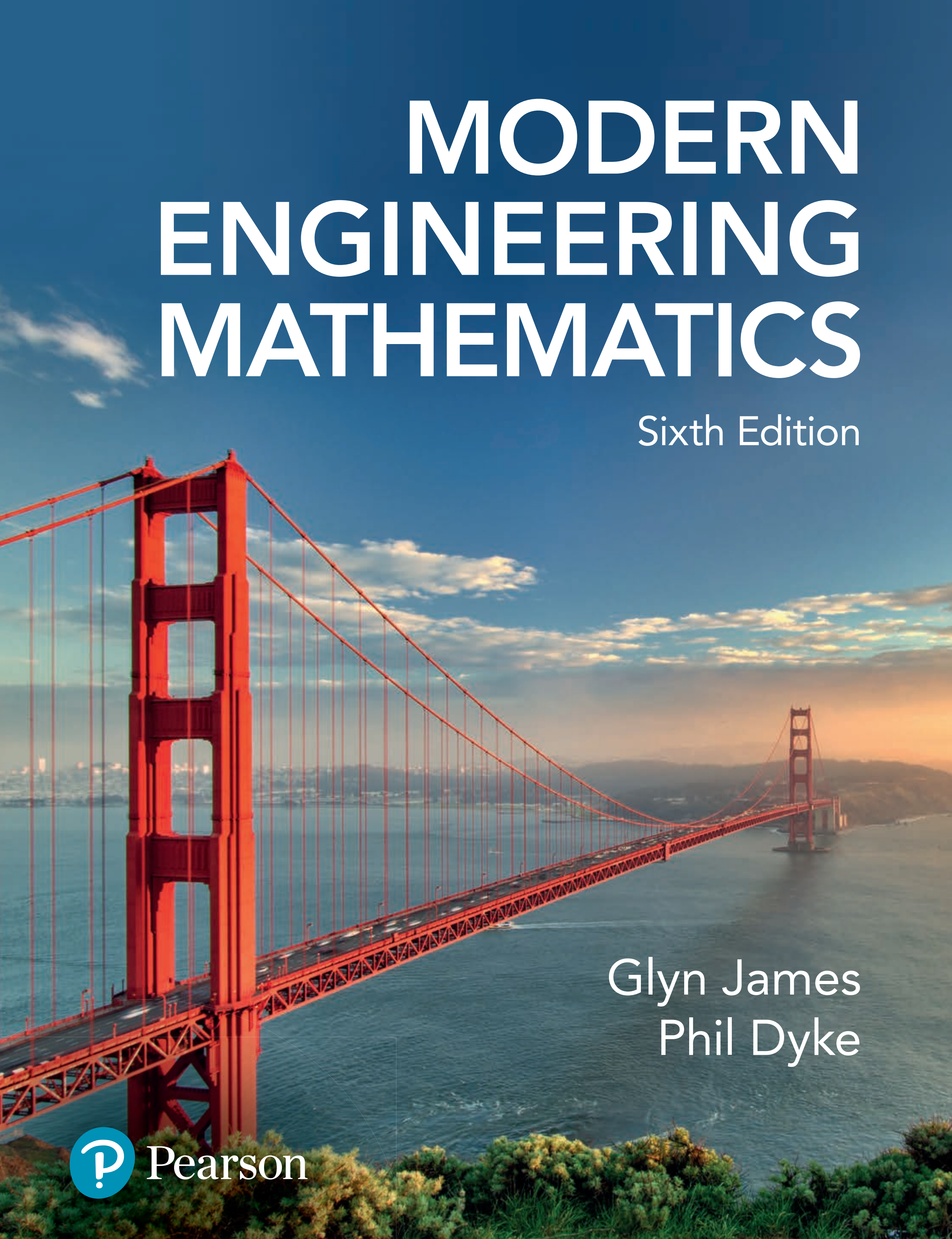

Pearson

Modern Engineering Mathematics

Solution (a) $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -j & 0 \\ 0 & j \end{bmatrix}$

so

$$AB + BA = 0$$

and the other two results follow similarly.

(b) From part (a)

$$AB - BA = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix} - \begin{bmatrix} -j & 0 \\ 0 & j \end{bmatrix} = \begin{bmatrix} 2j & 0 \\ 0 & -2j \end{bmatrix} = 2jC$$

and again the other two results follow similarly.

(c) These results can be obtained directly from part (a) since ***AB*** has already been calculated, similarly for ***BC*** and ***CA***.

Note: This example illustrates the use of matrices that have complex elements. Pauli discovered that the matrices ***A***, ***B*** and ***C*** have the properties (a), (b) and (c) required of the components of the spin of an electron.

Example 5.10

A rectangular site is to be levelled, and the amount of earth that needs to be removed must be determined. A survey of the site at a regular mesh of points 10 m apart is made. The heights in metres above the level required are given in the following table.

0	0.31	0.40	0.45	0.51	0.60
0.12	0.33	0.51	0.58	0.66	0.75
0.19	0.38	0.60	0.69	0.78	0.86
0.25	0.46	0.68	0.77	0.89	0.97

It is known that the approximate volume of a cell of side x and with corner heights of a, b, c and d is

$$V = \tfrac{1}{4}x^2(a + b + c + d)$$

Write the total approximate volume in matrix form and hence estimate the volume to be removed.

Solution Note that for the first row of cells the volume is

$$\begin{aligned} 25(0 &+ 0.31 \quad +0.31 + 0.40 \quad +0.40 + 0.45 \quad +0.45 + 0.51 \quad +0.51 + 0.60 \\ &+0.12 + 0.33 \quad +0.33 + 0.51 \quad +0.51 + 0.58 \quad +0.58 + 0.66 \quad +0.66 + 0.75) \end{aligned}$$

$$\begin{aligned} &= 25[0 + 2(0.31 + 0.40 + 0.45 + 0.51) + 0.60] \\ &\quad + 25[0.12 + 2(0.33 + 0.51 + 0.58 + 0.66) + 0.75] \end{aligned}$$

The second and third rows of cells are dealt with in a similar manner, so that, when we compute the total volume, we need to multiply the corner values by 1, the other side values by 2 and the centre values by 4. In matrix form this multiplication can be performed as

$$[1 \quad 2 \quad 2 \quad 1]\begin{bmatrix} 0 & 0.31 & 0.40 & 0.45 & 0.51 & 0.60 \\ 0.12 & 0.33 & 0.51 & 0.58 & 0.66 & 0.75 \\ 0.19 & 0.38 & 0.60 & 0.69 & 0.78 & 0.86 \\ 0.25 & 0.46 & 0.68 & 0.77 & 0.86 & 0.97 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

This can be checked by multiplying out the matrices. The checking can readily be done on one of the symbolic manipulation packages, such as the Symbolic Math Toolbox of MATLAB, by putting in general symbols for the matrix and verifying that, after the matrix multiplications, the elements are multiplied by the stated factors. Performing the calculation and multiplying by the 25 gives the total volume as 816.5 m^3.

A similar analysis can be applied to other situations – all that is needed is measured heights and a matrix multiplication routine on a computer to deal with the large amount of data that would be required. For other mesh shapes, or even irregular meshes, the method is similar, but the multiplying vectors will need careful calculation.

Example 5.11

A contractor makes two products P_1 and P_2. The four components required to make the products are subcontracted out and each of the components is made up from three ingredients A, B and C as follows:

Component	*Units of A*	*Units of B*	*Units of C*	*Make-up cost and profit for subcontractor*
1 requires	5	4	3	10
2 requires	2	1	1	7
3 requires	0	1	3	5
4 requires	3	4	1	2

The cost per unit of the ingredients A, B and C are a, b and c respectively. The contractor makes the product P_1 with 2 of component 1, 3 of component 2 and 4 of component 4, and the make-up cost is 15; product P_2 requires 1 of component 1, 1 of component 2, 1 of component 3 and 2 of component 4, and the make-up cost is 12. Find the cost to the contractor for P_1 and P_2. What is the change in costs if a increases to $(a + 1)$? It is found that the 5 units of A required for component 1 can be reduced to 4. What is the effect on the costs?

Solution

The information presented can be written naturally in matrix form. Let C_1, C_2, C_3 and C_4 be the cost the subcontractor charges the contractor for the four components, then the cost C_1 is computed as $C_1 = 5a + 4b + 3c + 10$. This expression is the first row of the matrix equation

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 10 \\ 7 \\ 5 \\ 2 \end{bmatrix}$$

and the other three costs follow in a similar manner. Now let p_1, p_2 be the costs of producing the final products. The costs are constructed in exactly the same way as

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} + \begin{bmatrix} 15 \\ 12 \end{bmatrix}$$

Substituting gives

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 & 3 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 2 & 3 & 0 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 7 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 15 \\ 12 \end{bmatrix}$$

or

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 28 & 27 & 13 \\ 13 & 14 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 64 \\ 38 \end{bmatrix}$$

Thus a simple matrix formulation gives a convenient way of coding the data. If a is increased to $(a + 1)$ then multiplying out shows that p_1 increases by 28 and p_2 by 13. If the 5 in the first matrix is reduced to 4 then the costs will be

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 26 & 27 & 13 \\ 12 & 14 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} 64 \\ 38 \end{bmatrix}$$

so p_1 is reduced by $2a$ and p_2 by a.

A similar approach can be used in more complicated, realistic situations. Storing and processing the information is convenient, particularly in conjunction with a computer package or spreadsheet.

Example 5.12

(a) Given the matrix $\boldsymbol{A} = \begin{bmatrix} \frac{3}{2} & -1 \\ 1 & -1 \end{bmatrix}$ verify that

$$\boldsymbol{A}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix} \quad \text{and} \quad \boldsymbol{A}\begin{bmatrix}1\\2\end{bmatrix}=\left(-\tfrac{1}{2}\right)\begin{bmatrix}1\\2\end{bmatrix}$$

and show that $2\boldsymbol{A}^2 = \boldsymbol{A} + \boldsymbol{I}$.

(b) By repeated application of this result show also that for any integer n

$$\boldsymbol{A}^n = \alpha\boldsymbol{A} + \beta\boldsymbol{I}$$

for some α, β.

Solution (a) The first two results follow by applying matrix multiplication; the importance of such results will be seen later in the section on eigenvalues. The next result follows since

$$\boldsymbol{A}^2 = \begin{bmatrix}\frac{3}{2} & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{3}{2} & -1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}\frac{5}{4} & -\frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix} \quad \text{and} \quad \boldsymbol{A}+\boldsymbol{I} = \begin{bmatrix}\frac{5}{2} & -1\\ 1 & 0\end{bmatrix}$$

and hence $2\boldsymbol{A}^2 = \boldsymbol{A} + \boldsymbol{I}$.

(b) To show the final result, note that multiplying by $2\boldsymbol{A}$ gives

$$4\boldsymbol{A}^3 = 2\boldsymbol{A}^2 + 2\boldsymbol{A} = (\boldsymbol{A} + \boldsymbol{I}) + 2\boldsymbol{A} = 3\boldsymbol{A} + \boldsymbol{I}$$

and repeating the process, multiplying by $2\boldsymbol{A}$

$$8\boldsymbol{A}^4 = 6\boldsymbol{A}^2 + 2\boldsymbol{A} = 3(\boldsymbol{A} + \boldsymbol{I}) + 2\boldsymbol{A} = 5\boldsymbol{A} + 3\boldsymbol{I}$$

The process of multiplying by $2\boldsymbol{A}$ and replacing $2\boldsymbol{A}^2$ by $(\boldsymbol{A} + \boldsymbol{I})$ can be applied repeatedly to give the final result.

Example 5.13 Find the values of x that make the matrix $\boldsymbol{Z}^5$ a diagonal matrix, where

$$\boldsymbol{Z} = \begin{bmatrix}x & 0 & 0\\ 0 & x & 1\\ 0 & -1 & 0\end{bmatrix}$$

Solution Although this problem can be done by hand it is tedious and a MATLAB solution is given.

Using MATLAB's Symbolic Math Toolbox the commands

```
syms x
Z = [x 0 0; 0 x 1; 0 -1 0];
Z5 = Z^5; simplify(Z5);
```

produce the matrix. The additional commands

```
solve(1 - 3*x^2 + x^4); double(ans)
```

produce the values

```
ans = 0.6180
     -1.6180
      1.6180
     -0.6180
```

5.2.7 Exercises

Check the answers to the exercises using MATLAB whenever possible.

19 Given the matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

evaluate where possible

$$\boldsymbol{AB}, \boldsymbol{BA}, \boldsymbol{BC}, \boldsymbol{CB}, \boldsymbol{CA}, \boldsymbol{AC}$$

20 For the matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) evaluate $(\boldsymbol{A} + \boldsymbol{B})^2$ and $\boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$

(b) evaluate $(\boldsymbol{A} + \boldsymbol{B})(\boldsymbol{A} - \boldsymbol{B})$ and $\boldsymbol{A}^2 - \boldsymbol{B}^2$

Repeat the calculations with the matrices

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 5 & 2 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 2 & -2 \\ -5 & 1 \end{bmatrix}$$

and explain the differences between the results for the two sets.

21 Show that for a square matrix $(\boldsymbol{A}^2)^{\mathrm{T}} = (\boldsymbol{A}^{\mathrm{T}})^2$.

22 Show that $\boldsymbol{AA}^{\mathrm{T}}$ is a symmetric matrix.

23 Find all the 2 × 2 matrices that commute (that is n, $\boldsymbol{AB} = \boldsymbol{BA}$) with $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$.

24 A matrix with m rows and n columns is said to be of type $m \times n$. Give simple examples of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to illustrate the following situations:

(a) $\boldsymbol{AB}$ is defined but $\boldsymbol{BA}$ is not;

(b) $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are both defined but have different type;

(c) $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are both defined and have the same type but are unequal.

25 Given

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

determine a symmetric matrix $\boldsymbol{C}$ and a skew-symmetric matrix $\boldsymbol{D}$ such that

$$\boldsymbol{A} = \boldsymbol{C} + \boldsymbol{D}$$

26 Given the matrices

$$\boldsymbol{a} = [3 \quad 2 \quad -1], \quad \boldsymbol{b} = \begin{bmatrix} 11 \\ 0 \\ 2 \end{bmatrix} \text{ and}$$

$$\boldsymbol{C} = \begin{bmatrix} 4 & 1 & 1 \\ -1 & 7 & -3 \\ -1 & 3 & 5 \end{bmatrix}$$

determine the elements of $\boldsymbol{G}$ where

$$(\boldsymbol{ab})\boldsymbol{I} + \boldsymbol{C}^2 = \boldsymbol{C}^{\mathrm{T}} + \boldsymbol{G}$$

and $\boldsymbol{I}$ is the unit matrix.

27 A firm allocates staff into four categories: welders, fitters, designers and administrators. It is estimated that for three main products the time spent, in hours, on each item is given in the following matrix.

	Boiler	Water tank	Holding frame
Welder	2	0.75	1.25
Fitter	1.4	0.5	1.75
Designer	0.3	0.1	0.1
Admin	0.1	0.25	0.3

The wages, pension contributions and overheads, in £ per hour, are known to be

	Welder	Fitter	Designer	Administrator
Wages	12	8	20	10
Pension	1	0.5	2	1
O/heads	0	0	1	3

Write the problem in matrix form and use matrix products to find the total cost of producing 10 boilers, 25 water tanks and 35 frames.

28 Given

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

evaluate $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$. Verify that

$$\boldsymbol{A}^3 - \boldsymbol{A}^2 - 3\boldsymbol{A} + \boldsymbol{I} = 0$$

29 Given

$$A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

show that

$$X^{T}AX = 27 \qquad \textbf{(5.6)}$$

implies that

$$5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3 = 27$$

Under the transformation

$$X = BY$$

show that (5.6) becomes

$$Y^{T}(B^{T}AB)Y = 27$$

If

$$B = \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

evaluate $B^{T}AB$, and hence show that

$$y_1^2 + 2y_2^2 + 3y_3^2 = 1$$

30 A well-known problem concerns a mythical country that has three cities, A, B and C, with a total population of 2400. At the end of each year it is decreed that all people must move to another city, half to one and half to the other. If a, b and c are the populations in the cities A, B and C respectively, show that in the next year the populations are given by

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Supposing that the three cities have initial populations of 600, 800 and 1000, what are the populations after ten years and after a very long time (a package such as MATLAB is ideal for the calculations)? (Note that this example is a version of a **Markov chain** problem. Markov chains have applications in many areas of science and engineering.)

31 Find values of h, k, l and m so that $A \neq 0$, $B \neq 0$, $A^2 = A$, $B^2 = B$ and $AB = 0$, where

$$A = h\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} k & -l & -l \\ -l & m & m \\ -l & m & m \end{bmatrix}$$

32 A computer screen has dimensions 20 cm × 30 cm. Axes are set up at the centre of the screen, as illustrated in Figure 5.5. A box containing an arrow has dimensions 2 cm × 2 cm and is situated with its centre at the point (−16, 10). It is first to be rotated through 45° in an anticlockwise direction. Find this transformation in the form

$$\begin{bmatrix} x' + 16 \\ y' - 10 \end{bmatrix} = A\begin{bmatrix} x + 16 \\ y - 10 \end{bmatrix}$$

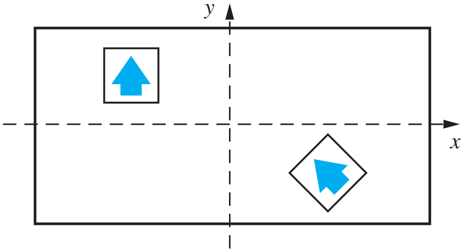

Figure 5.5 Manipulation of a computer screen in Question 32.

The rotated box is now moved to a new position with its centre at (16, −10). Find the overall transformation in the form

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + B\begin{bmatrix} x \\ y \end{bmatrix}$$

33 Given the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

it is known that $A^n = I$, the unit matrix, for some integer n; find this value.