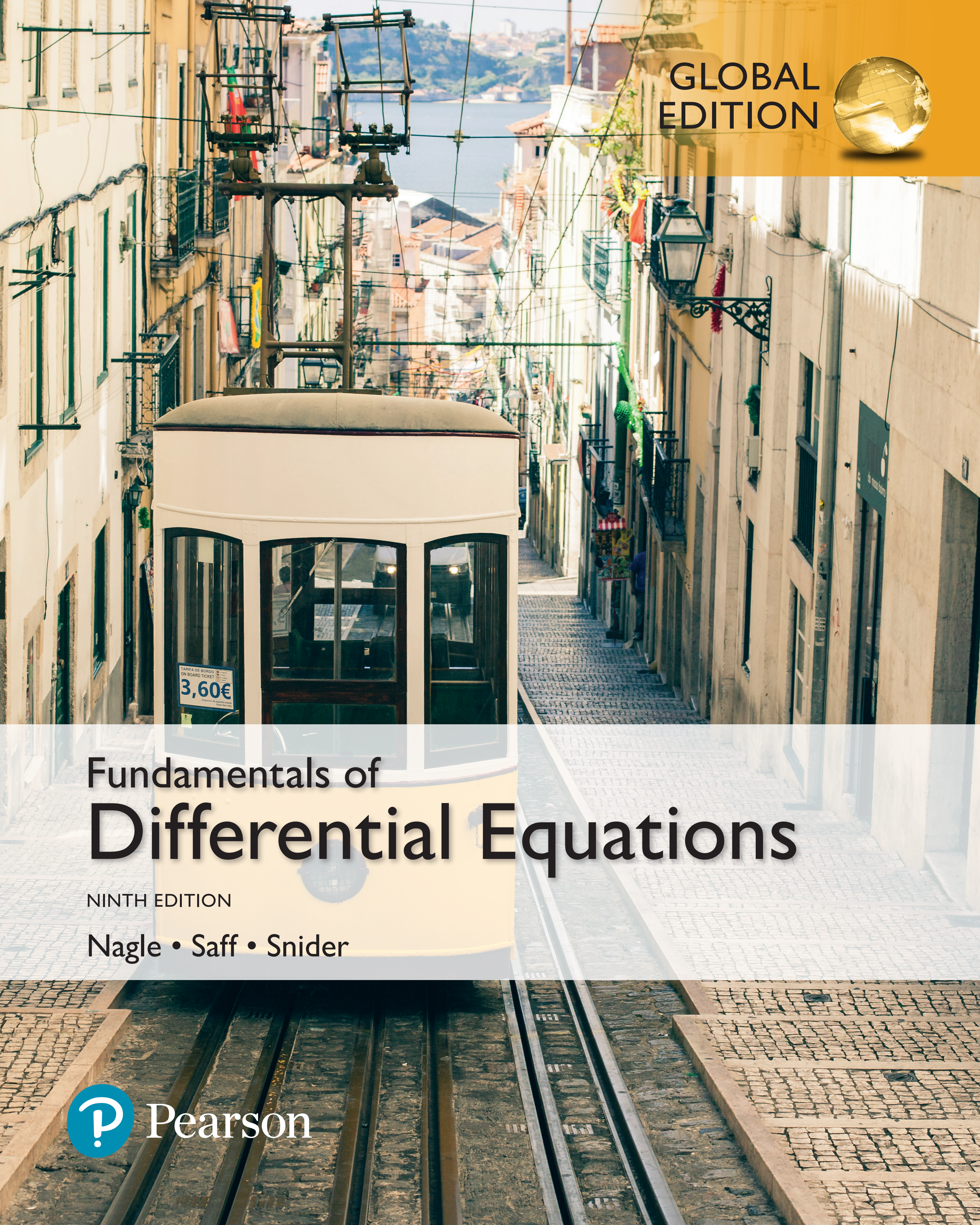

GLOBAL EDITION

Fundamentals of

Differential Equations

NINTH EDITION

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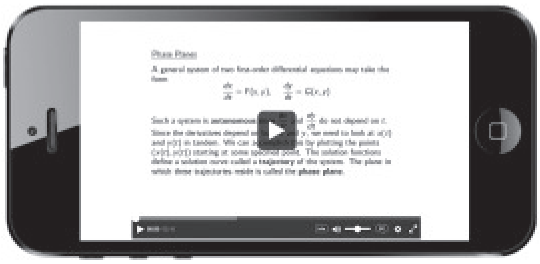

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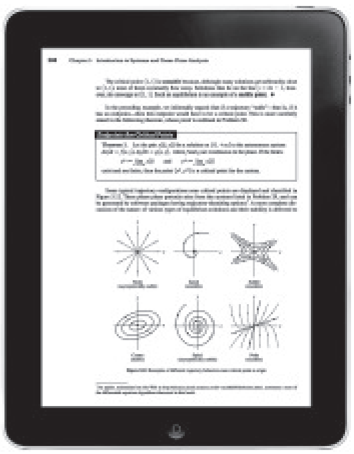

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Example 1 Determine the largest interval for which Theorem 5 ensures the existence and uniqueness of a solution to the initial value problem

$$(t-3)\frac{d^2y}{dt^2}+\frac{dy}{dt}+\sqrt{t}\,y=\ln t\,;\quad y(1)=3\,,\quad y'(1)=-5\,. \tag{5}$$

Solution The data $p(t)$, $q(t)$, and $g(t)$ in the standard form of the equation,

$$y''+py'+qy=\frac{d^2y}{dt^2}+\frac{1}{(t-3)}\frac{dy}{dt}+\frac{\sqrt{t}}{(t-3)}y=\frac{\ln t}{(t-3)}=g\,,$$

are simultaneously continuous in the intervals $0<t<3$ and $3<t<\infty$. The former contains the point $t_0=1$, where the initial conditions are specified, so Theorem 5 guarantees (5) has a unique solution in $0<t<3$. ◆

Theorem 5, embracing existence and uniqueness for the variable-coefficient case, is difficult to prove because we can't construct explicit solutions in the general case. So the proof is deferred to Chapter 13.† However, it is instructive to examine a special case that we can solve explicitly.

Cauchy–Euler, or Equidimensional, Equations

Definition 2. A linear second-order equation that can be expressed in the form

$$at^2y''(t)+bty'(t)+cy=f(t)\,, \tag{6}$$

where a, b, and c are constants, is called a **Cauchy–Euler,** or **equidimensional, equation.**

For example, the differential equation

$$3t^2y''+11ty'-3y=\sin t$$

is a Cauchy–Euler equation, whereas

$$2y''-3ty'+11y=3t-1$$

is *not* because the coefficient of y'' is 2, which is not a constant times t^2.

The nomenclature *equidimensional* comes about because if y has the dimensions of, say, meters and t has dimensions of time, then each term t^2y'', ty', and y has the same dimensions (meters). The coefficient of $y''(t)$ in (6) is at^2, and it is zero at $t=0$; equivalently, the standard form

$$y''+\frac{b}{at}y'+\frac{c}{at^2}y=\frac{f(t)}{at^2}$$

has discontinuous coefficients at $t=0$. Therefore, we can expect the solutions to be valid only for $t>0$ or $t<0$. Discontinuities in f, of course, will impose further restrictions.

†All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

To solve a *homogeneous* Cauchy–Euler equation, for $t > 0$, we exploit the equidimensional feature by looking for solutions of the form $y = t^r$, because then t^2y'', ty', and y each have the form *(constant)* $\times\, t^r$:

$$y = t^r\,,\quad ty' = trt^{r-1} = rt^r\,,\quad t^2y'' = t^2r(r-1)t^{r-2} = r(r-1)t^r\,,$$

and substitution into the homogeneous form of (6) (that is, with $g = 0$) yields a simple quadratic equation for r:

$$ar(r-1)t^r + brt^r + ct^r = [ar^2 + (b-a)r + c]t^r = 0\,,\quad \text{or}$$

(7) $$\boldsymbol{ar^2 + (b-a)r + c = 0\,,}$$

which we call the associated *characteristic equation.*

Example 2 Find two linearly independent solutions to the equation

$$3t^2y'' + 11ty' - 3y = 0\,,\quad t > 0\,.$$

Solution Inserting $y = t^r$ yields, according to (7),

$$3r^2 + (11-3)r - 3 = 3r^2 + 8r - 3 = 0\,,$$

whose roots $r = 1/3$ and $r = -3$ produce the independent solutions

$$y_1(t) = t^{1/3}\,,\quad y_2(t) = t^{-3}\quad (\text{for } t > 0)\,.\ \blacklozenge$$

Clearly, the substitution $y = t^r$ into a homogeneous *equidimensional* equation has the same simplifying effect as the insertion of $y = e^{rt}$ into the homogeneous *constant-coefficient* equation in Section 4.2. That means we will have to deal with the same encumbrances:

1. What to do when the roots of (7) are complex
2. What to do when the roots of (7) are equal

If r is complex, $r = \alpha + i\beta$, we can interpret $t^{\alpha+i\beta}$ by using the identity $t = e^{\ln t}$ and invoking Euler's formula [equation (5), Section 4.3]:

$$t^{\alpha+i\beta} = t^\alpha t^{i\beta} = t^\alpha e^{i\beta\ln t} = t^\alpha[\cos(\beta\ln t) + i\sin(\beta\ln t)]\,.$$

Then we simplify as in Section 4.3 by taking the real and imaginary parts to form independent solutions:

(8) $$\boldsymbol{y_1 = t^\alpha\cos(\beta\ln t)\,,\quad y_2 = t^\alpha\sin(\beta\ln t)\,.}$$

If r is a double root of the characteristic equation (7), then independent solutions of the Cauchy–Euler equation on $(0, \infty)$ are given by

(9) $$\boldsymbol{y_1 = t^r\,,\quad y_2 = t^r\ln t\,.}$$

This can be verified by direct substitution into the differential equation. Alternatively, the second, linearly independent, solution can be obtained by *reduction of order*, a procedure to be discussed shortly in Theorem 8. Furthermore, Problem 23 demonstrates that the substitution $t = e^x$ changes the homogeneous Cauchy–Euler equation into a homogeneous constant-coefficient equation, and the formats (8) and (9) then follow from our earlier deliberations.

We remark that if a homogeneous Cauchy–Euler equation is to be solved for $t < 0$, then one simply introduces the change of variable $t = -\tau$, where $\tau > 0$. The reader should verify via the chain rule that the identical characteristic equation (7) arises when $\tau^r = (-t)^r$ is substituted in the equation. Thus the solutions take the same form as (8), (9), but with t replaced by $-t$; for example, if r is a double root of (7), we get $(-t)^r$ and $(-t)^r \ln(-t)$ as two linearly independent solutions on $(-\infty, 0)$.

Example 3 Find a pair of linearly independent solutions to the following Cauchy–Euler equations for $t > 0$.

(a) $t^2y'' + 5ty' + 5y = 0$ **(b)** $t^2y'' + ty' = 0$

Solution For part (a), the characteristic equation becomes $r^2 + 4r + 5 = 0$, with the roots $r = -2 \pm i$, and (8) produces the real solutions $t^{-2}\cos(\ln t)$ and $t^{-2}\sin(\ln t)$.

For part (b), the characteristic equation becomes simply $r^2 = 0$ with the double root $r = 0$, and (9) yields the solutions $t^0 = 1$ and $\ln t$. ◆

In Chapter 8 we will see how one can obtain power series expansions for solutions to variable-coefficient equations when the coefficients are *analytic* functions. But, as we said, there is no procedure for explicitly solving the general case. Nonetheless, thanks to the existence/uniqueness result of Theorem 5, most of the other theorems and concepts of the preceding sections are easily extended to the variable-coefficient case, with the proviso that they apply only over intervals in which the governing functions $p(t), q(t), g(t)$ are continuous. Thus we have the following analog of Lemma 1, page 182.

A Condition for Linear Dependence of Solutions

Lemma 3. If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation

(10) $$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

on an interval *I* where the functions $p(t)$ and $q(t)$ are continuous and if the Wronskian†

$$W[y_1, y_2](t) := y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

is zero at *any* point t **of *I***, then y_1 and y_2 are linearly dependent **on *I***.

As in the constant-coefficient case, the Wronskian of two solutions is either identically zero or never zero on I, with the latter implying linear independence on I.

Precisely as in the proof for the constant-coefficient case, it can be verified that any linear combination $c_1y_1 + c_2y_2$ of solutions y_1 and y_2 to (10) is also a solution. In fact, these are the only solutions to (10) as stated in the following result.

Representation of Solutions to Initial Value Problems

Theorem 6. If $y_1(t)$ and $y_2(t)$ are any two solutions to the homogeneous differential equation (10) that are linearly independent on an interval I, then every solution to (10) on I is expressible as a linear combination of y_1 and y_2. Moreover, the initial value problem consisting of equation (10) and the initial conditions $y(t_0) = Y_0$, $y'(t_0) = Y_1$ has a unique solution on I for any point t_0 in I and any constants Y_0, Y_1.

---

†The determinant representation of the Wronskian was introduced in Problem 34, Section 4.2.

As in the constant-coefficient case, the linear combination $y_h = c_1y_1 + c_2y_2$ is called a **general solution** to (10) on I if y_1, y_2 are linearly independent solutions on I.

For the nonhomogeneous equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)\ , \tag{11}$$

a general solution on I **is given by** $y = y_p + y_h$, **where** $y_h = c_1y_1 + c_2y_2$ **is a general solution to the corresponding homogeneous equation** (10) **on** I **and** y_p **is a particular solution to** (11) **on** I. In other words, the solution to the initial value problem stated in Theorem 5 must be of this form for a suitable choice of the constants c_1, c_2. This follows, just as before, from a straightforward extension of the superposition principle for variable-coefficient equations described in Problem 30.

As illustrated at the end of the Section 4.6, if linearly independent solutions to the homogeneous equation (10) are known, then y_p can be determined for (11) by the variation of parameters method.

Variation of Parameters

Theorem 7. If y_1 and y_2 are two linearly independent solutions to the homogeneous equation (10) on an interval I where $p(t)$, $q(t)$, and $g(t)$ are continuous, then a particular solution to (11) is given by $y_p = v_1y_1 + v_2y_2$, where ${}_1$ and ${}_2$ are determined up to a constant by the pair of equations

$$y_1v_1' + y_2v_2' = 0\ ,$$
$$y_1'v_1' + y_2'v_2' = g\ ,$$

which have the solution

$$v_1(t) = \int \frac{-g(t)\ y_2(t)}{W[y_1, y_2](t)}\,dt\ , \quad v_2(t) = \int \frac{g(t)y_1(t)}{W[y_1, y_2](t)}\,dt\ . \tag{12}$$

Note the formulation (12) presumes that the differential equation has been put into standard form [that is, divided by $a_2(t)$].

The proofs of the constant-coefficient versions of these theorems in Sections 4.2 and 4.5 did not make use of the constant-coefficient property, so one can prove them in the general case by literally copying those proofs but interpreting the coefficients as variables. Unfortunately, however, there is no construction analogous to the method of undetermined coefficients for the variable-coefficient case.

What does all this mean? The only stumbling block for our completely solving nonhomogeneous initial value problems for equations with variable coefficients,

$$y'' + p(t)y' + q(t)y = g(t)\ ; \quad y(t_0) = Y_0\ \ , \quad y'(t_0) = Y_1\ ,$$

is the lack of an explicit procedure for constructing independent solutions to the associated homogeneous equation (10). If we *had* y_1 and y_2 as described in the variation of parameters formula, we could implement (12) to find y_p, formulate the general solution of (11) as $y_p + c_1y_1 + c_2y_2$, and (with the assurance that the Wronskian is nonzero) fit the constants to the initial conditions. But with the exception of the Cauchy–Euler equation and the ponderous power series machinery of Chapter 8, we are stymied at the outset; there is no general procedure for finding y_1 and y_2.

Ironically, we only need *one* nontrivial solution to the associated homogeneous equation, thanks to a procedure known as *reduction of order* that constructs a second, linearly independent solution y_2 from a known one y_1. So one might well feel that the following theorem rubs salt into the wound.

Reduction of Order

Theorem 8. If $y_1(t)$ is a solution, not identically zero, to the homogeneous differential equation (10) in an interval I (see page 217), then

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t)dt}}{y_1(t)^2}\,dt \tag{13}$$

is a second, linearly independent solution.

This remarkable formula can be confirmed directly, but the following derivation shows how the procedure got its name.

Proof of Theorem 8. Our strategy is similar to that used in the derivation of the variation of parameters formula, Section 4.6. Bearing in mind that cy_1 is a solution of (10) for any constant c, we replace c by a *function* $v(t)$ and propose the trial solution $y_2(t) = v(t)y_1(t)$, spawning the formulas

$$y_2' = vy_1' + v'y_1\,, \quad y_2'' = vy_1'' + 2v'y_1' + v''y_1\,.$$

Substituting these expressions into the differential equation (10) yields

$$(vy_1'' + 2v'y_1' + v''y_1) + p(vy_1' + v'y_1) + qvy_1 = 0\,,$$

or, on regrouping,

$$(y_1'' + py_1' + qy_1)v + y_1v'' + (2y_1' + py_1)v' = 0\,. \tag{14}$$

The group in front of the undifferentiated $v(t)$ is simply a copy of the left-hand member of the original differential equation (10), so it is zero.† Thus (14) reduces to

$$y_1v'' + (2y_1' + py_1)v' = 0\,, \tag{15}$$

which is actually a *first*-order equation in the variable $w \equiv v'$:

$$y_1w' + (2y_1' + py_1)w = 0\,. \tag{16}$$

Indeed, (16) is separable and can be solved immediately using the procedure of Section 2.2. Problem 48 on page 223 requests the reader to carry out the details of this procedure to complete the derivation of (13). ◆

Example 4 Given that $y_1(t) = t$ is a solution to

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0\,, \tag{17}$$

use the reduction of order procedure to determine a second linearly independent solution for $t > 0$.

---

†This is hardly a surprise; if were constant, y would be a solution with $v' = v'' = 0$ in (14).

Solution Rather than implementing the formula (13), let's apply the strategy used to derive it. We set $y_2(t) = v(t)y_1(t) = v(t)t$ and substitute $y_2' = v't + v$, $y_2'' = v''t + 2v'$ into (17) to find

$$v''t + 2v' - \frac{1}{t}(v't + v) + \frac{1}{t^2}vt = v''t + (2v' - v') = v''t + v' = 0\,. \tag{18}$$

As promised, (18) is a *separable first*-order equation in v', simplifying to $(v')'/(v') = -1/t$ with a solution $v' = 1/t$, or $v = \ln t$ (taking integration constants to be zero). Therefore, a second solution to (17) is $y_2 = vt = t\ln t$.

Of course (17) is a Cauchy–Euler equation for which (7) has equal roots:

$$ar^2 + (b - a)r + c = r^2 - 2r + 1 = (r - 1)^2 = 0\,,$$

and y_2 is precisely the form for the independent solution predicted by (9). ◆

Example 5 The following equation arises in the mathematical modeling of reverse osmosis.[†]

$$(\sin t)y'' - 2(\cos t)y' - (\sin t)y = 0\,, \quad 0 < t < \pi\,. \tag{19}$$

Find a general solution.

Solution As we indicated above, the tricky part is to find a single nontrivial solution. Inspection of (19) suggests that $y = \sin t$ or $y = \cos t$, combined with a little luck with trigonometric identities, might be solutions. In fact, trial and error shows that the cosine function works:

$$y_1 = \cos t\,, \quad y_1' = -\sin t\,, \quad y_1'' = -\cos t\,,$$
$$(\sin t)y_1'' - 2(\cos t)y_1' - (\sin t)y_1 = (\sin t)(-\cos t) - 2(\cos t)(-\sin t) - (\sin t)(\cos t) = 0\,.$$

Unfortunately, the sine function fails (try it).

So we use reduction of order to construct a second, independent solution. Setting $y_2(t) = v(t)y_1(t) = v(t)\cos t$ and computing $y_2' = v'\cos t - v\sin t$, $y_2'' = v''\cos t - 2v'\sin t - v\cos t$, we substitute into (19) to derive

$$(\sin t)[v''\cos t - 2v'\sin t - v\cos t] - 2(\cos t)[v'\cos t - v\sin t] - (\sin t)[v\cos t]$$
$$= v''(\sin t)(\cos t) - 2v'(\sin^2 t + \cos^2 t) = 0,$$

which is equivalent to the separated first-order equation

$$\frac{(v')'}{(v')} = \frac{2}{(\sin t)(\cos t)} = 2\frac{\sec^2 t}{\tan t}\,.$$

Taking integration constants to be zero yields $\ln v' = 2\ln(\tan t)$ or $v' = \tan^2 t$, and $v = \tan t - t$. Therefore, a second solution to (19) is $y_2 = (\tan t - t)\cos t = \sin t - t\cos t$. We conclude that a general solution is $c_1\cos t + c_2(\sin t - t\cos t)$. ◆

In this section we have seen that the *theory* for variable-coefficient equations differs only slightly from the constant-coefficient case (in that solution domains are restricted to intervals), but explicit solutions can be hard to come by. In the next section, we will supplement our exposition by describing some nonrigorous procedures that sometimes can be used to predict qualitative features of the solutions.

[†]Reverse osmosis is a process used to fortify the alcoholic content of wine, among other applications.