# ADVANCED MODERN ENGINEERING MATHEMATICS

Fifth Edition

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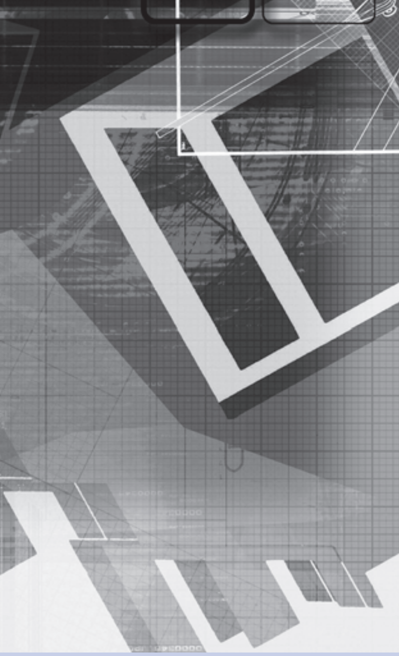
Pearson



# Advanced Modern Engineering Mathematics

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from which, using (4.24), it is readily deduced that

$$\frac{d}{dz}(\sinh z) = \cosh z$$

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We note from above that $e^z$ has the following real and imaginary parts:

$$\text{Re}(e^z) = e^x \cos y$$

$$\text{Im}(e^z) = e^x \sin y$$

In real variables the exponential and circular functions are contrasted, one being monotonic, the other oscillatory. In complex variables, however, the real and imaginary parts of $e^z$ are (two-variable) combinations of exponential and circular functions, which might seem surprising for an exponential function. Similarly, the circular functions of a complex variable have unfamiliar properties. For example, it is easy to see that $|\cos z|$ and $|\sin z|$ are unbounded for complex $z$ by using the above relationships between circular and hyperbolic functions of complex variables. Contrast this with $|\cos x| \leqslant 1$ and $|\sin x| \leqslant 1$ for a real variable $x$.

In a similar way to the method adopted in Examples 4.9 and 4.10 it can be shown that the derivatives of the majority of functions $f(x)$ of a real variable $x$ carry over to the complex variable case $f(z)$ at points where $f(z)$ is analytic. Thus, for example,

$$\frac{d}{dz} z^n = nz^{n-1}$$

for all $z$ in the $z$ plane, and

$$\frac{d}{dz} \ln z = \frac{1}{z}$$

for all $z$ in the $z$ plane except for points on the non-positive real axis, where $\ln z$ is non-analytic.

It can also be shown that the rules associated with derivatives of a function of a real variable, such as the sum, product, quotient and chain rules, carry over to the complex variable case. Thus,

$$\frac{d}{dz}[f(z) + g(z)] = \frac{df(z)}{dz} + \frac{dg(z)}{dz}$$

$$\frac{d}{dz}[f(z)\,g(z)] = f(z)\frac{dg(z)}{dz} + \frac{df(z)}{dz}g(z)$$

$$\frac{d}{dz} f(g(z)) = \frac{df}{dg}\frac{dg}{dz}$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)\,g'(z)}{[g(z)]^2}$$

### 4.3.2 Conjugate and harmonic functions

A pair of functions $u(x, y)$ and $v(x, y)$ of the real variables $x$ and $y$ that satisfy the Cauchy–Riemann equations (4.19) are said to be **conjugate functions**. (Note here the different use of the word 'conjugate' to that used in complex number work, where $z^* = x - jy$ is the complex conjugate of $z = x + jy$.) Conjugate functions satisfy the orthogonality property in that the curves in the $(x, y)$ plane defined by $u(x, y) =$ constant and $v(x, y) =$ constant are orthogonal curves. This follows since the gradient at any point on the curve $u(x, y) =$ constant is given by

$$\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_u = -\frac{\partial u}{\partial y}\bigg/\frac{\partial u}{\partial x}$$

and the gradient at any point on the curve $v(x, y) =$ constant is given by

$$\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_v = -\frac{\partial v}{\partial y}\bigg/\frac{\partial v}{\partial x}$$

It follows from the Cauchy–Riemann equations (4.19) that

$$\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_u \left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_v = -1$$

so the curves are orthogonal.

A function that satisfies the Laplace equation in two dimensions is said to be **harmonic**; that is, $u(x, y)$ is a harmonic function if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

It is readily shown (see Example 4.12) that if $f(z) = u(x, y) + jv(x, y)$ is analytic, so that the Cauchy–Riemann equations are satisfied, then both $u$ and $v$ are **harmonic** functions. Therefore $u$ and $v$ are **conjugate harmonic functions**. Harmonic functions have applications in such areas as stress analysis in plates, inviscid two-dimensional fluid flow and electrostatics.

**Example 4.11**

Given $u(x, y) = x^2 - y^2 + 2x$, find the conjugate function $v(x, y)$ such that $f(z) = u(x, y) + jv(x, y)$ is an analytic function of $z$ throughout the $z$ plane.

**Solution**

We are given $u(x, y) = x^2 - y^2 + 2x$, and, since $f(z) = u + jv$ is to be analytic, the Cauchy–Riemann equations must hold. Thus, from (4.19),

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + 2$$

Integrating this with respect to $y$ gives

$$v = 2xy + 2y + F(x)$$

where $F(x)$ is an arbitrary function of $x$, since the integration was performed holding $x$ constant. Differentiating $v$ partially with respect to $x$ gives

$$\frac{\partial v}{\partial x} = 2y + \frac{\mathrm{d}F}{\mathrm{d}x}$$

but this equals $-\partial u/\partial y$ by the second of the Cauchy–Riemann equations (4.19). Hence

$$\frac{\partial u}{\partial y} = -2y - \frac{\mathrm{d}F}{\mathrm{d}x}$$

But since $u = x^2 - y^2 + 2x$, $\partial u/\partial y = -2y$, and comparison yields $F(x) = $ constant. This constant is set equal to zero, since no conditions have been given by which it can be determined. Hence

$$u(x, y) + \mathrm{j}v(x, y) = x^2 - y^2 + 2x + \mathrm{j}(2xy + 2y)$$

To confirm that this is a function of $z$, note that $f(z)$ is $f(x + \mathrm{j}y)$, and becomes just $f(x)$ if we set $y = 0$. Therefore we set $y = 0$ to obtain

$$f(x + \mathrm{j}0) = f(x) = u(x, 0) + \mathrm{j}v(x, 0) = x^2 + 2x$$

and it follows that

$$f(z) = z^2 + 2z$$

which can be easily checked by separation into real and imaginary parts.

**Example 4.12**

Show that the real and imaginary parts $u(x, y)$ and $v(x, y)$ of a complex analytic function $f(z)$ are harmonic.

**Solution**

Since

$$f(z) = u(x, y) + \mathrm{j}v(x, y)$$

is analytic, the Cauchy–Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

are satisfied. Differentiating the first with respect to $x$ gives

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

which is $-\partial^2 v/\partial y^2$, by the second Cauchy–Riemann equation. Hence

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}, \quad \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and $v$ is a harmonic function.

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = -\frac{\partial^2 u}{\partial x^2}$$

so that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and $u$ is also a harmonic function. We have assumed that both $u$ and $v$ have continuous second-order partial derivatives, so that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}, \qquad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

## 4.3.3 Exercises

24 Determine whether the following functions are analytic, and find the derivative where appropriate:

(a) $z\,\mathrm{e}^z$ (b) $\sin 4z$

(c) $zz^*$ (d) $\cos 2z$

25 Determine the constants $a$ and $b$ in order that

$$w = x^2 + ay^2 - 2xy + \mathrm{j}(bx^2 - y^2 + 2xy)$$

be analytic. For these values of $a$ and $b$ find the derivative of $w$, and express both $w$ and $\mathrm{d}w/\mathrm{d}z$ as functions of $z = x + \mathrm{j}y$.

26 Find a function $v(x, y)$ such that, given $u = 2x(1 - y)$, $f(z) = u + \mathrm{j}v$ is analytic in $z$.

27 Show that $\phi(x, y) = \mathrm{e}^x(x\cos y - y\sin y)$ is a harmonic function, and find the conjugate harmonic function $\psi(x, y)$. Write $\phi(x, y) + \mathrm{j}\psi(x, y)$ as a function of $z = x + \mathrm{j}y$ only.

28 Show that $u(x, y) = \sin x \cosh y$ is harmonic. Find the harmonic conjugate $v(x, y)$ and express $w = u + \mathrm{j}v$ as a function of $z = x + \mathrm{j}y$.

29 Find the orthogonal trajectories of the following families of curves:

(a) $x^3y - xy^3 = \alpha$ (constant $\alpha$)

(b) $\mathrm{e}^{-x}\cos y + xy = \alpha$ (constant $\alpha$)

30 Find the real and imaginary parts of the functions

(a) $z^2\,\mathrm{e}^{2z}$

(b) $\sin 2z$

Verify that they are analytic and find their derivatives.

31 Give a definition of the inverse sine function $\sin^{-1} z$ for complex $z$. Find the real and imaginary parts of $\sin^{-1} z$. (*Hint*: Put $z = \sin w$, split into real and imaginary parts, and with $w = u + \mathrm{j}v$ and $z = x + \mathrm{j}y$ solve for $u$ and $v$ in terms of $x$ and $y$.) Is $\sin^{-1} z$ analytic? If so, what is its derivative?

32 Establish that if $z = x + \mathrm{j}y$, $|\sinh y| \leq |\sin z| \leq \cosh y$.

## 4.3.4 Mappings revisited

In Section 4.2 we examined mappings from the $z$ plane to the $w$ plane, where in the main the relationship between $w$ and $z$, $w = f(z)$ was linear or bilinear. There is an important property of mappings, hinted at in Example 4.8 when considering the mapping $w = z^2$. A mapping $w = f(z)$ that preserves angles is called **conformal**. Under such a mapping, the angle between two intersecting curves in the $z$ plane is the same as the angle between the corresponding intersecting curves in the $w$ plane. The sense of the angle is also preserved. That is, if $\theta$ is the angle between curves 1 and 2 taken in the anticlockwise sense in the $z$ plane then $\theta$ is also the angle between the image of curve 1 and the image of curve 2 in the $w$ plane, and it too is taken in the anticlockwise sense.

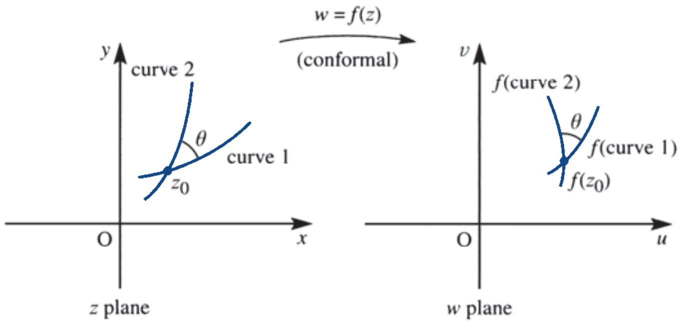


**Figure 4.15** Conformal mappings.

Figure 4.15 should make the idea of a conformal mapping clearer. If $f(z)$ is analytic then $w = f(z)$ defines a conformal mapping except at points where the derivative $f'(z)$ is zero.

Clearly the linear mappings

$$w = \alpha z + \beta \quad (\alpha \neq 0)$$

are conformal everywhere, since $\mathrm{d}w/\mathrm{d}z = \alpha$ and is not zero for any point in the $z$ plane. Bilinear mappings given by (4.12) are not so straightforward to check. However, as we saw in Section 4.2.4, (4.12) can be rearranged as

$$w = \lambda + \frac{\mu}{\alpha z + \beta} \quad (\alpha, \mu \neq 0)$$

Thus

$$\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{\mu\alpha}{(\alpha z + \beta)^2}$$

which again is never zero for any point in the $z$ plane. In fact, the only mapping we have considered so far that has a point at which it is not conformal everywhere is $w = z^2$ (cf. Example 4.8), which is not conformal at $z = 0$.

**Example 4.13**

Determine the points at which the mapping $w = z + 1/z$ is not conformal and demonstrate this by considering the image in the $w$ plane of the real axis in the $z$ plane.

**Solution**

Taking $z = x + \mathrm{j}y$ and $w = u + \mathrm{j}v$, we have

$$w = u + \mathrm{j}v = x + \mathrm{j}y + \frac{x - \mathrm{j}y}{x^2 + y^2}$$

which, on equating real and imaginary parts, gives

$$u = x + \frac{x}{x^2 + y^2}$$

$$v = y - \frac{y}{x^2 + y^2}$$

The real axis, $y = 0$, in the $z$ plane corresponds to $v = 0$, the real axis in the $w$ plane. Note, however, that the fixed point of the mapping is given by

$$z = z + \frac{1}{z}$$

or $z = \infty$. From the Cauchy–Riemann equations it is readily shown that $w$ is analytic everywhere except at $z = 0$. Also, $\mathrm{d}w/\mathrm{d}z = 0$ when

$$1 - \frac{1}{z^2} = 0, \quad \text{that is} \quad z = \pm 1$$

which are both on the real axis. Thus the mapping fails to be conformal at $z = 0$ and $z = \pm 1$. The image of $z = 1$ is $w = 2$, and the image of $z = -1$ is $w = -2$. Consideration of the image of the real axis is therefore perfectly adequate, since this is a curve passing through each point where $w = z + 1/z$ fails to be conformal. It would be satisfying if we could analyse this mapping in the same manner as we did with $w = z^2$ in Example 4.8. Unfortunately, we cannot do this, because the algebra gets unwieldy (and, indeed, our knowledge of algebraic curves is also too scanty). Instead, let us look at the image of the point $z = 1 + \varepsilon$, where $\varepsilon$ is a small real number. $\varepsilon > 0$ corresponds to the point Q just to the right of $z = 1$ on the real axis in the $z$ plane, and the point P just to the left of $z = 1$ corresponds to $\varepsilon < 0$ (Figure 4.16).

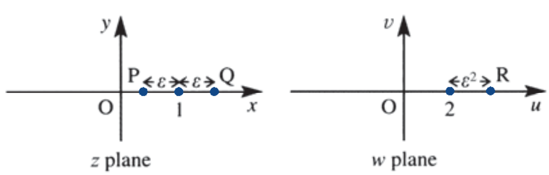


**Figure 4.16** Image of $z = 1 + \varepsilon$ of Example 4.13.

If $z = 1 + \varepsilon$ then

$$\begin{aligned} w &= 1 + \varepsilon + \frac{1}{1 + \varepsilon} \\ &= 1 + \varepsilon + (1 + \varepsilon)^{-1} \\ &= 1 + \varepsilon + 1 - \varepsilon + \varepsilon^2 - \varepsilon^3 + \cdots \\ &\simeq 2 + \varepsilon^2 \end{aligned}$$

if $|\varepsilon|$ is much smaller than 1 (we shall discuss the validity of the power series expansion in Section 4.4). Whether $\varepsilon$ is positive or negative, the point $w = 2 + \varepsilon^2$ is to the right of $w = 2$ in the $w$ plane as indicated by the point R in Figure 4.16. Therefore, as $\varepsilon \to 0$, a curve (the real axis) that passes through $z = 1$ in the $z$ plane making an angle $\theta = \pi$ corresponds to a curve (again the real axis) that approaches $w = 2$ in the $w$ plane along the real axis from the right making an angle $\theta = 0$. Non-conformality has thus been confirmed. The treatment of $z = -1$ follows in an identical fashion, so the details are omitted. Note that when $y = 0$ ($v = 0$), $u = x + 1/x$ so, as the real axis in the $z$ plane is traversed from $x = -\infty$ to $x = 0$, the real axis in the $w$ plane is traversed from