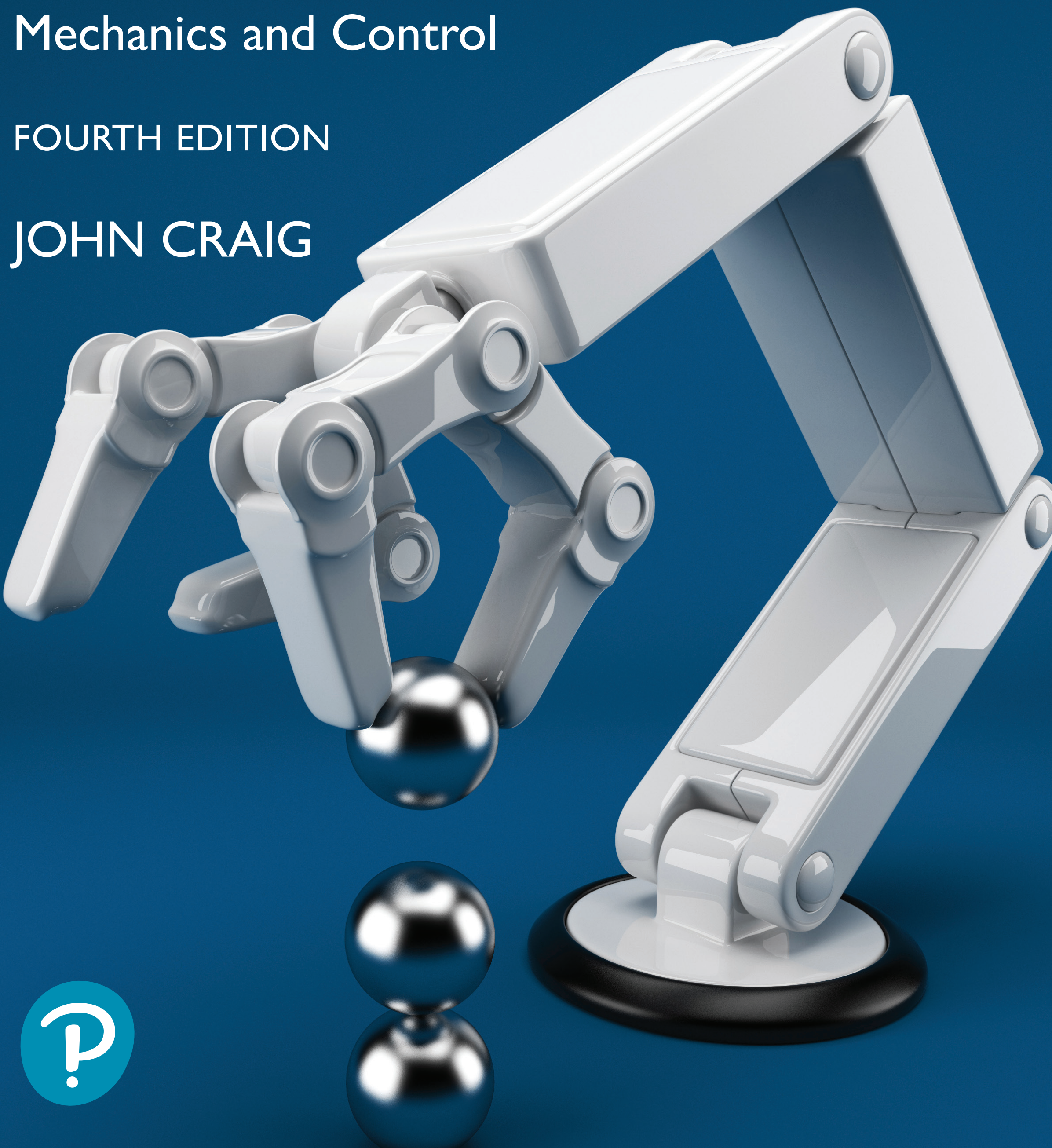
GLOBAL EDITION

Introduction to ROBOTICS

Mechanics and Control

FOURTH EDITION

JOHN CRAIG



Introduction to Robotics

Mechanics and Control

Fourth Edition
Global Edition

John J. Craig

Pearson

To solve an equation of this form, we make the trigonometric substitutions

$$p_x = \rho \cos\phi, \text{ and}$$
$$p_y = \rho \sin\phi, \tag{4.58}$$

where

$$\rho = \sqrt{p_x^2 + p_y^2},$$
$$\phi = \text{Atan2}(p_y, p_x). \tag{4.59}$$

Substituting (4.58) into (4.57), we obtain

$$c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho}. \tag{4.60}$$

From the difference-of-angles formula,

$$\sin(\phi - \theta_1) = \frac{d_3}{\rho}. \tag{4.61}$$

Hence,

$$\cos(\phi - \theta_1) = \pm\sqrt{1 - \frac{d_3^2}{\rho^2}}, \tag{4.62}$$

and so

$$\phi - \theta_1 = \text{Atan2}\left(\frac{d_3}{\rho}, \pm\sqrt{1 - \frac{d_3^2}{\rho^2}}\right). \tag{4.63}$$

Finally, the solution for θ_1 may be written as

$$\theta_1 = \text{Atan2}(p_y, p_x) - \text{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right). \tag{4.64}$$

Note that we have found two possible solutions for θ_1, corresponding to the plus-or-minus sign in (4.64). Now that θ_1 is known, the left-hand side of (4.56) is known. If we equate both the (1,4) elements and the (3,4) elements from the two sides of (4.56), we obtain

$$c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2,$$
$$-p_x = a_3 s_{23} + d_4 c_{23} + a_2 s_2. \tag{4.65}$$

If we square equations (4.65) and (4.57) and add the resulting equations, we obtain

$$a_3 c_3 - d_4 s_3 = K, \tag{4.66}$$

where

$$K = \frac{p_x^2 + p_y^2 + p_x^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}. \tag{4.67}$$

Note that dependence on θ_1 has been removed from (4.66). Equation (4.66) is of the same form as (4.57), and so can be solved by the same kind of trigonometric substitution to yield a solution for θ_3:

$$\theta_3 = \mathrm{Atan2}(a_3, d_4) - \mathrm{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}). \tag{4.68}$$

The plus-or-minus sign in (4.68) leads to two different solutions for θ_3. If we consider (4.54) again, we can now rewrite it so that all the left-hand side is a function of only knowns and θ_2:

$$[{}^0_3T(\theta_2)]^{-1}\, {}^0_6T = {}^3_4T(\theta_4)\, {}^4_5T(\theta_5)\, {}^5_6T(\theta_6), \tag{4.69}$$

or

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T, \tag{4.70}$$

where ${}^3_6T$ is given by equation (3.11) developed in Chapter 3. Equating both the (1,4) elements and the (2,4) elements from the two sides of (4.70), we get

$$\begin{aligned} c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 &= a_3, \\ -c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 &= d_4. \end{aligned} \tag{4.71}$$

These equations can be solved simultaneously for s_{23} and c_{23}, resulting in

$$\begin{aligned} s_{23} &= \frac{(-a_3 - a_2c_3)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4)}{p_z^2 + (c_1p_x + s_1p_y)^2}, \\ c_{23} &= \frac{(a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)}{p_z^2 + (c_1p_x + s_1p_y)^2}. \end{aligned} \tag{4.72}$$

The denominators are equal and positive, so we solve for the sum of θ_2 and θ_3 as

$$\begin{aligned} \theta_{23} = \mathrm{Atan2}[&(-a_3 - a_2c_3)p_z - (c_1p_x + s_1p_y)(d_4 - a_2s_3), \\ &(a_2s_3 - d_4)p_z - (a_3 + a_2c_3)(c_1p_x + s_1p_y)]. \end{aligned} \tag{4.73}$$

Equation (4.73) computes four values of θ_{23}, according to the four possible combinations of solutions for θ_1 and θ_3; then, four possible solutions for θ_2 are computed as

$$\theta_2 = \theta_{23} - \theta_3, \tag{4.74}$$

where the appropriate solution for θ_3 is used when forming the difference.

Now, the entire left side of (4.70) is known. Equating both the (1,3) elements and the (3,3) elements from the two sides of (4.70), we get

$$\begin{aligned} r_{13}c_1c_{23} + r_{23}s_1c_{23} - r_{33}s_{23} &= -c_4s_5, \\ -r_{13}s_1 + r_{23}c_1 &= s_4s_5. \end{aligned} \tag{4.75}$$

As long as $s_5 \neq 0$, we can solve for θ_4 as

$$\theta_4 = \mathrm{Atan2}(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}). \tag{4.76}$$

When $\theta_5 = 0$, the manipulator is in a singular configuration in which joint axes 4 and 6 line up and cause the same motion of the last link of the robot. In this case, all that matters (and all that can be solved for) is the sum or difference of θ_4 and θ_6. This situation is detected by checking whether both arguments of the Atan2 in (4.76) are near zero. If so, θ_4 is chosen arbitrarily,[4] and when θ_6 is computed later, it will be computed accordingly.

If we consider (4.54) again, we can now rewrite it so all the left-hand side is a function of only knowns and θ_4, by rewriting it as

$$[{}^0_4T(\theta_4)]^{-1}\,{}^0_6T = {}^4_5T(\theta_5)\,{}^5_6T(\theta_6), \tag{4.77}$$

where $[{}^0_4T(\theta_4)]^{-1}$ is given by

$$\begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.78}$$

and ${}^4_6T$ is given by equation (3.10) developed in Chapter 3. Equating both the (1,3) elements and the (3,3) elements from the two sides of (4.77), we get

$$\begin{aligned} r_{13}(c_1c_{23}c_4 + s_1s_4) + r_{23}(s_1c_{23}c_4 - c_1s_4) - r_{33}(s_{23}c_4) &= -s_5, \\ r_{13}(-c_1s_{23}) + r_{23}(-s_1s_{23}) + r_{33}(-c_{23}) &= c_5. \end{aligned} \tag{4.79}$$

Hence, we can solve for θ_5 as

$$\theta_5 = \text{Atan2}(s_5, c_5), \tag{4.80}$$

where s_5 and c_5 are given by (4.79).

Applying the same method one more time, we compute $({}^0_5T)^{-1}$ and write (4.54) in the form

$$({}^0_5T)^{-1}\,{}^0_6T = {}^5_6T(\theta_6). \tag{4.81}$$

Equating both the (3,1) elements and the (1,1) elements from the two sides of (4.77) as we have done before, we get

$$\theta_6 = \text{Atan2}(s_6, c_6), \tag{4.82}$$

where

$$\begin{aligned} s_6 &= -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4), \\ c_6 &= r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5] \\ &\quad - r_{31}(s_{23}c_4c_5 + c_{23}s_5). \end{aligned}$$

Because of the plus-or-minus signs appearing in (4.64) and (4.68), these equations compute four solutions. Additionally, there are four more solutions obtained by

[4] It is usually chosen to be equal to the present value of joint 4.

"flipping" the wrist of the manipulator. For each of the four solutions computed above, we obtain the flipped solution by

$$\begin{aligned}\theta_4' &= \theta_4 + 180^\circ,\\ \theta_5' &= -\theta_5, \\ \theta_6' &= \theta_6 + 180^\circ.\end{aligned} \tag{4.83}$$

After all eight solutions have been computed, some (or even all) of them might have to be discarded because of joint-limit violations. Of any remaining valid solutions, usually the one closest to the present manipulator configuration is chosen.

The Yasukawa Motoman L-3

As the second example, we will solve the kinematic equations of the Yasukawa Motoman L-3, which were developed in Chapter 3. This solution will be partially algebraic, and partially geometric. The Motoman L-3 has three features that make the inverse kinematic problem quite different from that of the PUMA. First, the manipulator has only five joints, so it is not able to position and orient its end-effector in order to attain *general* goal frames. Second, the four-bar type of linkages and chain-drive scheme cause one actuator to move two or more joints. Third, the actuator position limits are not constants, but depend on the positions of the other actuators, so finding out whether a computed set of actuator values is in range is not trivial.

If we consider the nature of the subspace of the Motoman manipulator (and the same applies to many manipulators with five degrees of freedom), we quickly realize that this subspace can be described by giving one constraint on the attainable orientation: The pointing direction of the tool, that is, the $\hat{Z}_T$ axis, must lie in the "plane of the arm." This plane is the vertical plane that contains the axis of joint 1 and the point where axes 4 and 5 intersect. The orientation nearest to a general orientation is the one obtained by rotating the tool's pointing direction so it lies in the plane, using a minimum amount of rotation. Without developing an explicit expression for this subspace, we will construct a method for projecting a general goal frame into it. Note that this entire discussion is for the case that the wrist frame and tool frame differ only by a translation along $\hat{Z}_w$.

In Fig. 4.9, we indicate the plane of the arm by its normal, $\hat{M}$, and the desired pointing direction of the tool by $\hat{Z}_T$. This pointing direction must be rotated by angle θ about some vector $\hat{K}$ in order to cause the new pointing direction, $\hat{Z}_T'$, to lie in the plane. It is clear that the $\hat{K}$ that minimizes θ lies in the plane, and is orthogonal to both $\hat{Z}_T$ and $\hat{Z}_T'$.

For any given goal frame, $\hat{M}$ is defined as

$$\hat{M} = \frac{1}{\sqrt{p_x^2 + p_y^2}} \begin{bmatrix} -p_y \\ p_x \\ 0 \end{bmatrix}, \tag{4.84}$$

where p_x and p_y are the X and Y coordinates of the desired tool position. Then, K is given by

$$K = \hat{M} \times \hat{Z}_T. \tag{4.85}$$

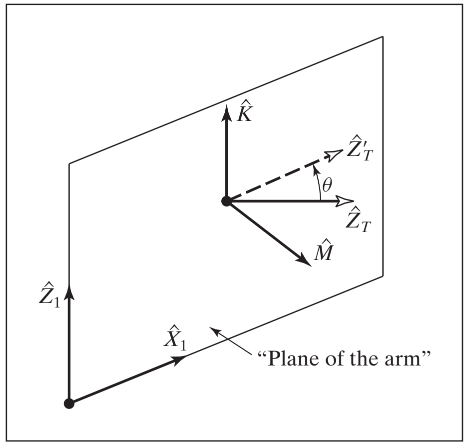


FIGURE 4.9: Rotating a goal frame into the Motoman's subspace.

The new $\hat{Z}'_T$ is

$$\hat{Z}'_T = \hat{K} \times \hat{M}. \tag{4.86}$$

The amount of rotation, θ, is given by

$$\cos\theta = \hat{Z}_T \cdot \hat{Z}'_T,$$
$$\sin\theta = (\hat{Z}_T \times \hat{Z}'_T) \cdot \hat{K}. \tag{4.87}$$

Using Rodriques's formula (see Exercise 2.20), we have

$$\hat{Y}'_T = c\theta \hat{Y}_T + s\theta(\hat{K} \times \hat{Y}_T) + (1 - c\theta)(\hat{K} \cdot \hat{Y}_T)\hat{K}. \tag{4.88}$$

Finally, we compute the remaining unknown column of the new rotation matrix of the tool as

$$\hat{X}'_T = \hat{Y}'_T \times \hat{Z}'_T. \tag{4.89}$$

Equations (4.84) through (4.89) describe a method of projecting a given general goal orientation into the subspace of the Motoman robot.

Assuming that the given wrist frame, ${}^B_W T$, lies in the manipulator's subspace, we solve the kinematic equations as follows. In deriving the kinematic equations for the Motoman L-3, we formed the product of link transformations:

$${}^0_5T = {}^0_1T\, {}^1_2T\, {}^2_3T\, {}^3_4T\, {}^4_5T. \tag{4.90}$$

If we let

$${}^0_5T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.91}$$

and premultiply both sides by ${}^0_1T^{-1}$, we have

$${}^0_1T^{-1}\, {}^0_5T = {}^1_2T\, {}^2_3T\, {}^3_4T\, {}^4_5T, \tag{4.92}$$

where the left-hand side is

$$\begin{bmatrix} c_1r_{11}+s_1r_{21} & c_1r_{12}+s_1r_{22} & c_1r_{13}+s_1r_{23} & c_1p_x+s_1p_y \\ -r_{31} & -r_{32} & -r_{33} & -p_z \\ -s_1r_{11}+c_1r_{21} & -s_1r_{12}+c_1r_{22} & -s_1r_{13}+c_1r_{23} & -s_1p_x+c_1p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.93}$$

and the right-hand side is

$$\begin{bmatrix} * & * & s_{234} & * \\ * & * & -c_{234} & * \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \tag{4.94}$$

in the latter, several of the elements have not been shown. Equating the (3,4) elements, we get

$$-s_1p_x + c_1p_y = 0, \tag{4.95}$$

which gives us[5]

$$\theta_1 = \text{Atan2}(p_y, p_x). \tag{4.96}$$

Equating the (3,1) and (3,2) elements, we get

$$\begin{aligned} s_5 &= -s_1r_{11} + c_1r_{21}, \\ c_5 &= -s_1r_{12} + c_1r_{22}, \end{aligned} \tag{4.97}$$

from which we calculate θ_5 as

$$\theta_5 = \text{Atan2}(r_{21}c_1 - r_{11}s_1, r_{22}c_1 - r_{12}s_1). \tag{4.98}$$

Equating the (2,3) and (1,3) elements, we get

$$\begin{aligned} c_{234} &= r_{33}, \\ s_{234} &= c_1r_{13} + s_1r_{23}, \end{aligned} \tag{4.99}$$

which leads to

$$\theta_{234} = \text{Atan2}(r_{13}c_1 + r_{23}s_1, r_{33}). \tag{4.100}$$

To solve for the individual angles θ_2, θ_3, and θ_4, we will take a geometric approach. Figure 4.10 shows the plane of the arm with point A at joint axis 2, point B at joint axis 3, and point C at joint axis 4.

From the law of cosines applied to triangle ABC, we have

$$\cos\theta_3 = \frac{p_x^2 + p_y^2 + p_z^2 - l_2^2 - l_3^2}{2l_2l_3}. \tag{4.101}$$

Next, we have[6]

$$\theta_3 = \text{Atan2}\left(\sqrt{1-\cos^2\theta_3}, \cos\theta_3\right). \tag{4.102}$$

[5]For this manipulator, a second solution would violate joint limits, and so is not calculated.

[6]For this manipulator, a second solution would violate joint limits, and so is not calculated.