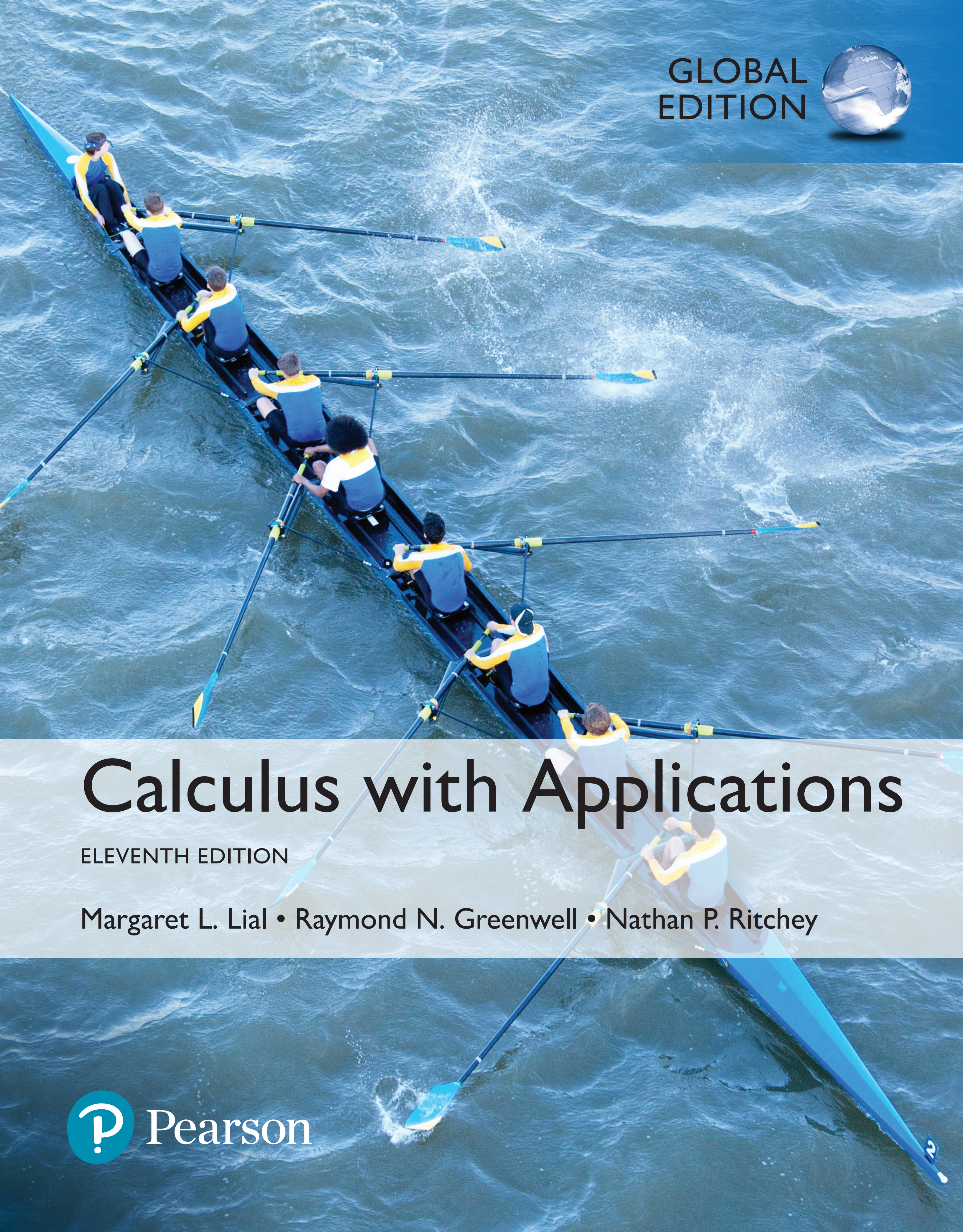

GLOBAL EDITION

Calculus with Applications

ELEVENTH EDITION

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Pearson

Calculus with Applications

4 Calculating the Derivative

By differentiating the function defining a mathematical model we can see how the model's output changes with the input. In an exercise in Section 2 we explore a rational-function model for the length of the rest period needed to recover from vigorous exercise such as riding a bike. The derivative indicates how the rest required changes with the work expended in kilocalories per minute.

In the previous chapter, we found the derivative to be a useful tool for describing the rate of change, velocity, and the slope of a curve. Taking the derivative by using the definition, however, can be difficult. To take full advantage of the power of the derivative, we need faster ways of calculating the derivative. That is the goal of this chapter.

4.1 Techniques for Finding Derivatives

APPLY IT **How can a manager determine the best production level if the relationship between profit and production is known? How fast is the number of Americans who are expected to be over 100 years old growing?**

These questions can be answered by finding the derivative of an appropriate function. We shall return to them at the end of this section in Examples 9 and 10.

Using the definition to calculate the derivative of a function is a very involved process even for simple functions. In this section we develop rules that make the calculation of derivatives much easier. Keep in mind that even though the process of finding a derivative will be greatly simplified with these rules, *the interpretation of the derivative will not change*. But first, a few words about notation are in order.

In addition to $f'(x)$, there are several other commonly used notations for the derivative.

Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)], \quad \text{or} \quad D_x[f(x)].$$

The dy/dx notation for the derivative (read "the derivative of y with respect to x") is sometimes referred to as *Leibniz notation,* named after one of the co-inventors of calculus, Gottfried Wilhelm von Leibniz (1646–1716). (The other was Sir Isaac Newton, 1642–1727.)

With the above notation, the derivative of $y = f(x) = 2x^3 + 4x$, for example, which was found in Example 5 of Section 3.4 to be $f'(x) = 6x^2 + 4$, would be written

$$\frac{dy}{dx} = 6x^2 + 4$$

$$\frac{d}{dx}(2x^3 + 4x) = 6x^2 + 4$$

$$D_x(2x^3 + 4x) = 6x^2 + 4.$$

A variable other than x is often used as the independent variable. For example, if $y = f(t)$ gives population growth as a function of time, then the derivative of y with respect to t could be written

$$f'(t), \quad \frac{dy}{dt}, \quad \frac{d}{dt}[f(t)], \quad \text{or} \quad D_t[f(t)].$$

Other variables also may be used to name the function, as in $g(x)$ or $h(t)$.

Now we will use the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

to develop some rules for finding derivatives more easily than by the four-step process given in the previous chapter.

The first rule tells how to find the derivative of a constant function defined by $f(x) = k$, where k is a constant real number. Since $f(x + h)$ is also k, by definition $f'(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0,$$

establishing the following rule.

Constant Rule

If $f(x) = k$, where k is any real number, then

$$f'(x) = \frac{d[k]}{dx} = 0.$$

(The derivative of a constant is 0.)

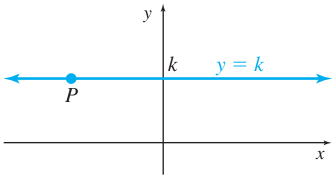

FIGURE 1

This rule is logical because the derivative represents rate of change, and a constant function, by definition, does not change. Figure 1 illustrates this constant rule geometrically; it shows a graph of the horizontal line $y = k$. At any point P on this line, the tangent line at P is the line itself. Since a horizontal line has a slope of 0, the slope of the tangent line is 0. This agrees with the result above: The derivative of a constant is 0.

EXAMPLE 1 Derivative of a Constant

(a) If $f(x) = 9$, then $f'(x) = 0$.

(b) If $h(t) = \pi$, then $D_t[h(t)] = 0$.

(c) If $y = 2^3$, then $dy/dx = 0$.

Functions of the form $y = x^n$, where n is a fixed real number, are very common in applications. To obtain a rule for finding the derivative of such a function, we can use the definition to work out the derivatives for various special values of n. This was done in Section 3.4 in Example 4 to show that for $f(x) = x^2$, $f'(x) = 2x$.

For $f(x) = x^3$, the derivative is found as follows.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

The binomial theorem (discussed in most intermediate and college algebra texts) was used to expand $(x + h)^3$ in the last step. Now, the limit can be determined.

$$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$
$$= 3x^2$$

The results in the table on the next page were found in a similar way, using the definition of the derivative. (These results are modifications of some of the examples and exercises from the previous chapter.)

Derivative of $f(x) = x^n$		
Function	n	**Derivative**
$f(x) = x$	1	$f'(x) = 1 = 1x^0$
$f(x) = x^2$	2	$f'(x) = 2x = 2x^1$
$f(x) = x^3$	3	$f'(x) = 3x^2$
$f(x) = x^4$	4	$f'(x) = 4x^3$
$f(x) = x^{-1}$	-1	$f'(x) = -1 \cdot x^{-2} = \frac{-1}{x^2}$
$f(x) = x^{1/2}$	1/2	$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$

These results suggest the following rule.

Power Rule

If $f(x) = x^n$ for any real number n, then

$$f'(x) = \frac{d[x^n]}{dx} = nx^{n-1}.$$

(The derivative of $f(x) = x^n$ is found by multiplying by the exponent n and decreasing the exponent on x by 1.)

While the power rule is true for every real-number value of n, a proof is given here only for positive integer values of n. This proof follows the steps used above in finding the derivative of $f(x) = x^3$.

For any real numbers p and q, by the binomial theorem,

$$(p + q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{2}p^{n-2}q^2 + \cdots + npq^{n-1} + q^n.$$

Replacing p with x and q with h gives

$$(x + h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n,$$

from which

$$(x + h)^n - x^n = nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n.$$

Dividing each term by h yields

$$\frac{(x + h)^n - x^n}{h} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}.$$

Use the definition of derivative, and the fact that each term except the first contains h as a factor and thus approaches 0 as h approaches 0, to get

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x + h)^n - x^n}{h} \\ &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}0 + \cdots + nx0^{n-2} + 0^{n-1} \\ &= nx^{n-1}. \end{aligned}$$

This shows that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$, proving the power rule for positive integer values of n.

EXAMPLE 2 Power Rule

(a) If $f(x) = x^6$, find $f'(x)$.

SOLUTION $f'(x) = 6x^{6-1} = 6x^5$

(b) If $y = t = t^1$, find $\dfrac{dy}{dt}$.

SOLUTION $\dfrac{dy}{dt} = 1t^{1-1} = t^0 = 1$

FOR REVIEW

At this point you may wish to turn back to Sections R.6 and R.7 for a review of negative exponents and rational exponents. The relationship between powers, roots, and rational exponents is explained there.

(c) If $y = 1/x^3$, find dy/dx.

SOLUTION Use a negative exponent to rewrite this equation as $y = x^{-3}$; then

$$\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} \quad \text{or} \quad \frac{-3}{x^4}.$$

(d) Find $D_x(x^{4/3})$.

SOLUTION $D_x(x^{4/3}) = \dfrac{4}{3}x^{4/3-1} = \dfrac{4}{3}x^{1/3}$

(e) If $y = \sqrt{z}$, find dy/dz.

SOLUTION Rewrite this as $y = z^{1/2}$; then

$$\frac{dy}{dz} = \frac{1}{2}z^{1/2-1} = \frac{1}{2}z^{-1/2} \quad \text{or} \quad \frac{1}{2z^{1/2}} \quad \text{or} \quad \frac{1}{2\sqrt{z}}.$$

YOUR TURN 1 If $f(t) = \dfrac{1}{\sqrt{t}}$, find $f'(t)$.

TRY YOUR TURN 1

The next rule shows how to find the derivative of the product of a constant and a function.

Constant Times a Function

Let k be a real number. If $g'(x)$ exists, then the derivative of $f(x) = k \cdot g(x)$ is

$$f'(x) = \frac{d}{dx}[k \cdot g(x)] = k \cdot g'(x).$$

(The derivative of a constant times a function is the constant times the derivative of the function.)

This rule is proved with the definition of the derivative and rules for limits.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{kg(x+h) - kg(x)}{h} \\ &= \lim_{h\to 0} k\frac{[g(x+h) - g(x)]}{h} && \text{Factor out } k. \\ &= k\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} && \text{Limit rule 1} \\ &= k \cdot g'(x) && \text{Definition of derivative} \end{aligned}$$

EXAMPLE 3 Derivative of a Constant Times a Function

(a) If $y = 8x^4$, find $\dfrac{dy}{dx}$.

SOLUTION $\dfrac{dy}{dx} = 8(4x^3) = 32x^3$

(b) If $y = -\frac{3}{4}x^{12}$, find dy/dx.

SOLUTION $\frac{dy}{dx} = -\frac{3}{4}(12x^{11}) = -9x^{11}$

(c) Find $D_t(-8t)$.

SOLUTION $D_t(-8t) = -8(1) = -8$

(d) Find $D_p(10p^{3/2})$.

SOLUTION $D_p(10p^{3/2}) = 10\left(\frac{3}{2}p^{1/2}\right) = 15p^{1/2}$

(e) If $y = \frac{6}{x}$, find $\frac{dy}{dx}$.

SOLUTION Rewrite this as $y = 6x^{-1}$; then

$$\frac{dy}{dx} = 6(-1x^{-2}) = -6x^{-2} \quad \text{or} \quad \frac{-6}{x^2}.$$

TRY YOUR TURN 2

YOUR TURN 2 If $y = 3\sqrt{x}$, find dy/dx.

EXAMPLE 4 Beagles

Researchers have determined that the daily energy requirements of female beagles who are at least 1 year old change with respect to age according to the function

$$E(t) = 753t^{-0.1321},$$

where $E(t)$ is the daily energy requirements (in kJ/W$^{0.67}$) for a dog that is t years old. ***Source: Journal of Nutrition***.

(a) Find $E'(t)$.

SOLUTION Using the rules of differentiation we find that

$$E'(t) = 753(-0.1321)t^{-0.1321-1} = -99.4713t^{-1.1321}.$$

(b) Determine the rate of change of the daily energy requirements of a 2-year-old female beagle.

SOLUTION $E'(2) = -99.4713(2)^{-1.1321} \approx -45.4$

Thus, the daily energy requirements of a 2-year-old female beagle are decreasing at the rate of 45.4 kJ/W$^{0.67}$ per year.

The final rule in this section is for the derivative of a function that is a sum or difference of terms.

Sum or Difference Rule

If $f(x) = u(x) \pm v(x)$, and if $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = \frac{d}{dx}[u(x) \pm v(x)] = u'(x) \pm v'(x).$$

(The derivative of a sum or difference of functions is the sum or difference of the derivatives.)