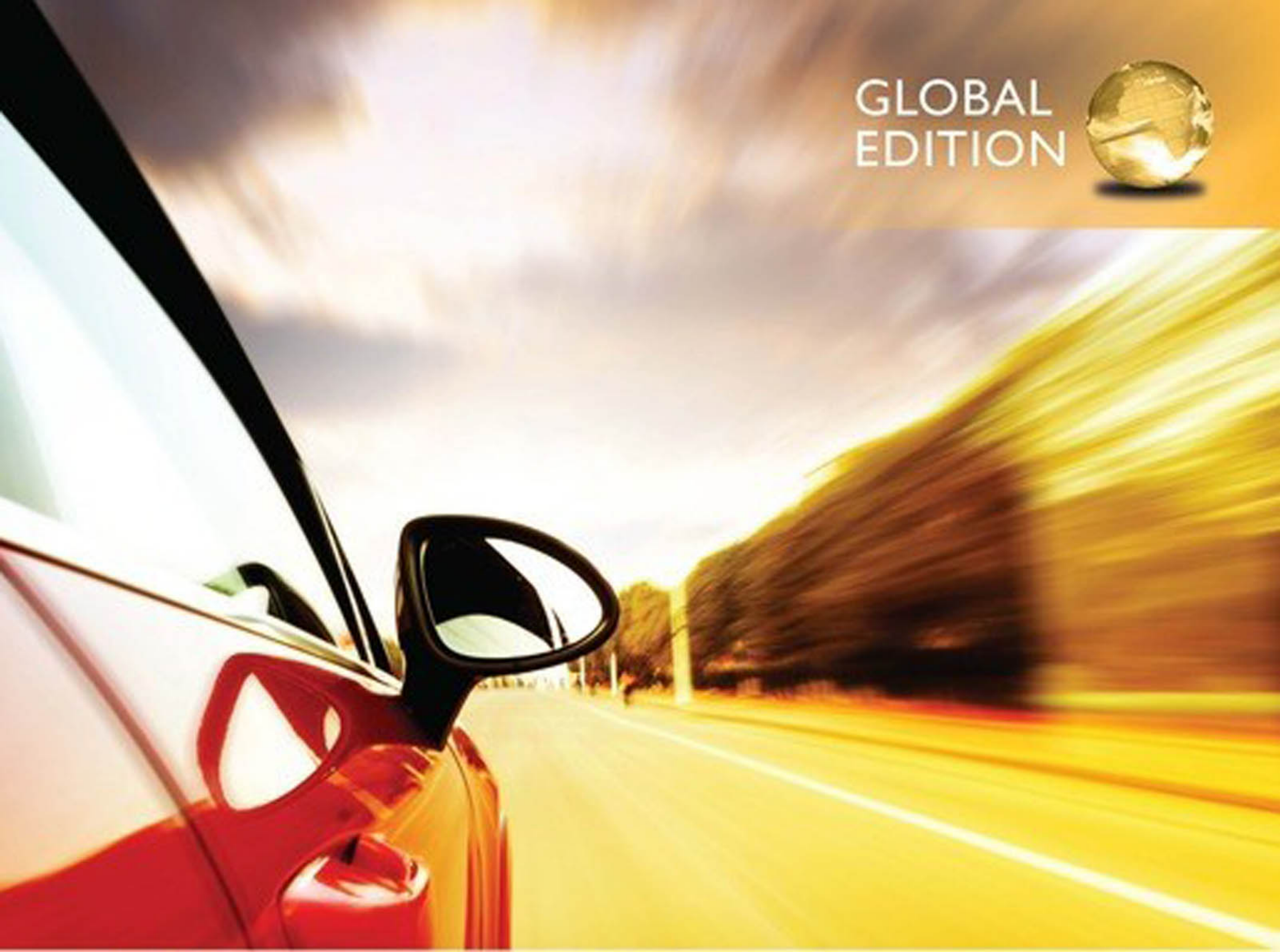

GLOBAL EDITION

Differential Equations and Boundary Value Problems

Computing and Modeling

FIFTH EDITION

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ALWAYS LEARNING

PEARSON

Table of Integrals

ELEMENTARY FORMS

1. $\int u\,dv = uv - \int v\,du$

2. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C \quad \text{if } n \neq -1$

3. $\int \frac{du}{u} = \ln|u| + C$

4. $\int e^u\,du = e^u + C$

5. $\int a^u\,du = \frac{a^u}{\ln a} + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \sec^2 u\,du = \tan u + C$

9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u\,\tan u\,du = \sec u + C$

11. $\int \csc u\,\cot u\,du = -\csc u + C$

12. $\int \tan u\,du = \ln|\sec u| + C$

13. $\int \cot u\,du = \ln|\sin u| + C$

14. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

15. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

17. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

18. $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

TRIGONOMETRIC FORMS

19. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

20. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

21. $\int \tan^2 u\,du = \tan u - u + C$

22. $\int \cot^2 u\,du = -\cot u - u + C$

23. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$

24. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$

25. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

26. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

27. $\int \sec^3 u\,du = \frac{1}{2}\sec u\,\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

28. $\int \csc^3 u\,du = -\frac{1}{2}\csc u\,\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

29. $\int \sin au\,\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C \quad \text{if } a^2 \neq b^2$

(*Continued on Rear Endpaper*)

are the *critical buckling forces* of the rod. Only when the compressive force P is one of these critical forces should the rod "buckle" out of its straight (undeflected) shape. The smallest compressive force for which this occurs is

$$P_1 = \frac{\pi^2 EI}{L^2}. \tag{29}$$

This smallest critical force P_1 is called the *Euler buckling force* for the rod; it is the upper bound for those compressive forces to which the rod can safely be subjected without buckling. (In practice a rod may fail at a significantly smaller force due to a contribution of factors not taken into account by the mathematical model discussed here.)

Example 7 For instance, suppose that we want to compute the Euler buckling force for a steel rod 10 ft long having a circular cross section 1 in. in diameter. In cgs units we have

$$E = 2 \times 10^{12} \text{ g/cm·s}^2,$$

$$L = (10 \text{ ft})\left(30.48 \frac{\text{cm}}{\text{ft}}\right) = 304.8 \text{ cm}, \quad \text{and}$$

$$I = \frac{\pi}{4}\left[(0.5 \text{ in.})\left(2.54 \frac{\text{cm}}{\text{in.}}\right)\right]^4 \approx 2.04 \text{ cm}^4.$$

Upon substituting these values in Eq. (29) we find that the critical force for this rod is

$$P_1 \approx 4.34 \times 10^8 \text{ dyn} \approx 976 \text{ lb},$$

using the conversion factor 4.448×10^5 dyn/lb. ■

3.8 Problems

The eigenvalues in Problems 1 through 5 are all nonnegative. First determine whether $\lambda = 0$ is an eigenvalue; then find the positive eigenvalues and associated eigenfunctions.

1. $y'' + \lambda y = 0$; $y'(0) = 0$, $y(1) = 0$
2. $y'' + \lambda y = 0$; $y'(0) = 0$, $y'(\pi) = 0$
3. $y'' + \lambda y = 0$; $y(-\pi) = 0$, $y(\pi) = 0$
4. $y'' + \lambda y = 0$; $y'(-\pi) = 0$, $y'(\pi) = 0$
5. $y'' + \lambda y = 0$; $y(-2) = 0$, $y'(2) = 0$
6. Consider the eigenvalue problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, \quad y(1) + y'(1) = 0.$$

All the eigenvalues are nonnegative, so write $\lambda = \alpha^2$ where $\alpha \geqq 0$. **(a)** Show that $\lambda = 0$ is not an eigenvalue. **(b)** Show that $y = A\cos\alpha x + B\sin\alpha x$ satisfies the endpoint conditions if and only if $B = 0$ and α is a positive root of the equation $\tan z = 1/z$. These roots $\{\alpha_n\}_1^\infty$ are the abscissas of the points of intersection of the curves $y = \tan z$ and $y = 1/z$, as indicated in Fig. 3.8.13. Thus the eigenvalues and eigenfunctions of this problem are the numbers $\{\alpha_n^2\}_1^\infty$ and the functions $\{\cos\alpha_n x\}_1^\infty$, respectively.

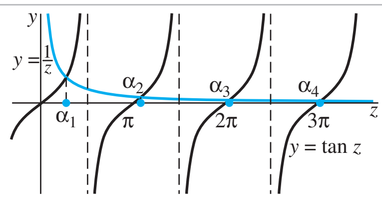

FIGURE 3.8.13. The eigenvalues are determined by the intersections of the graphs of $y = \tan z$ and $y = 1/z$ (Problem 6).

7. Consider the eigenvalue problem

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) + y'(1) = 0;$$

all its eigenvalues are nonnegative. **(a)** Show that $\lambda = 0$ is not an eigenvalue. **(b)** Show that the eigenfunctions are the functions $\{\sin\alpha_n x\}_1^\infty$, where α_n is the nth positive root of the equation $\tan z = -z$. **(c)** Draw a sketch indicating the roots $\{\alpha_n\}_1^\infty$ as the points of intersection of the curves $y = \tan z$ and $y = -z$. Deduce from this sketch that $\alpha_n \approx (2n-1)\pi/2$ when n is large.
8. Consider the eigenvalue problem

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) = y'(1);$$

all its eigenvalues are nonnegative. **(a)** Show that $\lambda = 0$ is an eigenvalue with associated eigenfunction $y_0(x) = x$.

(b) Show that the remaining eigenfunctions are given by $y_n(x) = \sin \beta_n x$, where β_n is the nth positive root of the equation $\tan z = z$. Draw a sketch showing these roots. Deduce from this sketch that $\beta_n \approx (2n+1)\pi/2$ when n is large.

9. Prove that the eigenvalue problem of Example 4 has no negative eigenvalues.

10. Prove that the eigenvalue problem

$$y'' + \lambda y = 0; \quad y'(0) = 0, \quad y(1) = 0$$

has no negative eigenvalues.

11. Prove that the eigenvalue problem

$$y'' + \lambda y = 0; \quad y(0) = 0, \quad y(1) = y'(1)$$

has no negative eigenvalues. (Suggestion: Show that the only root of the equation $\tanh z = z$ is $z = 0$.)

12. Consider the eigenvalue problem

$$y'' + \lambda y = 0; \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi),$$

which is not of the type in (10) because the two endpoint conditions are not "separated" between the two endpoints. **(a)** Show that $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) \equiv 1$. **(b)** Show that there are no negative eigenvalues. **(c)** Show that the nth positive eigenvalue is n^2 and that it has two linearly independent associated eigenfunctions, $\cos nx$ and $\sin nx$.

13. Consider the eigenvalue problem

$$y'' - 2y' + \lambda y = 0; \quad y(0) = y(\pi) = 0.$$

(a) Show that $\lambda = 1$ is not an eigenvalue. **(b)** Show that there is no eigenvalue λ such that $\lambda < 1$. **(c)** Show that the nth positive eigenvalue is $\lambda_n = n^2 + 1$, with associated eigenfunction $y_n(x) = e^x \sin nx$.

14. Consider the eigenvalue problem

$$y'' - 2y' + \lambda y = 0; \quad y(0) = 0, \quad y'(\pi) = 0.$$

Show that the eigenvalues are all positive and that the nth positive eigenvalue is $\lambda_n = \alpha_n^2 + 1$ with associated eigenfunction $y_n(x) = e^x \sin \alpha_n x$, where α_n is the nth positive root of $\tan \pi z = -z$.

15. **(a)** A uniform cantilever beam is fixed at $x = 0$ and free at its other end, where $x = L$. Show that its shape is given by

$$y(x) = \frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2).$$

(b) Show that $y'(x) = 0$ only at $x = 0$, and thus that it follows (why?) that the maximum deflection of the cantilever is $y_{max} = y(L) = wL^4/(8EI)$.

16. **(a)** Suppose that a beam is fixed at its ends $x = 0$ and $x = L$. Show that its shape is given by

$$y(x) = \frac{w}{24EI}(x^4 - 2Lx^3 + L^2x^2).$$

(b) Show that the roots of $y'(x) = 0$ are $x = 0$, $x = L$, and $x = L/2$, so it follows (why?) that the maximum deflection of the beam is

$$y_{max} = y\left(\frac{L}{2}\right) = \frac{wL^4}{384EI},$$

one-fifth that of a beam with simply supported ends.

17. For the simply supported beam whose deflection curve is given by Eq. (24), show that the only root of $y'(x) = 0$ in $[0, L]$ is $x = L/2$, so it follows (why?) that the maximum deflection is indeed that given in Eq. (25).

18. **(a)** A beam is fixed at its left end $x = 0$ but is simply supported at the other end $x = L$. Show that its deflection curve is

$$y(x) = \frac{w}{48EI}(2x^4 - 5Lx^3 + 3L^2x^2).$$

(b) Show that its maximum deflection occurs where $x = \left(15 - \sqrt{33}\right)L/16$ and is about 41.6% of the maximum deflection that would occur if the beam were simply supported at each end.

4 Introduction to Systems of Differential Equations

4.1 First-Order Systems and Applications

In the preceding chapters we have discussed methods for solving an ordinary differential equation that involves only one dependent variable. Many applications, however, require the use of two or more dependent variables, each a function of a single independent variable (typically time). Such a problem leads naturally to a *system* of simultaneous ordinary differential equations. We will usually denote the independent variable by t and the dependent variables (the unknown functions of t) by $x_1, x_2, x_3, \ldots$ or by $x, y, z, \ldots$. Primes will indicate derivatives with respect to t.

We will restrict our attention to systems in which the number of equations is the same as the number of dependent variables (unknown functions). For instance, a system of two first-order equations in the dependent variables x and y has the general form

$$\begin{aligned} f(t, x, y, x', y') &= 0, \\ g(t, x, y, x', y') &= 0, \end{aligned} \tag{1}$$

where the functions f and g are given. A **solution** of this system is a pair $x(t)$, $y(t)$ of functions of t that satisfy both equations identically over some interval of values of t.

For an example of a second-order system, consider a particle of mass m that moves in space under the influence of a force field $\mathbf{F}$ that depends on time t, the position $(x(t), y(t), z(t))$ of the particle, and its velocity $(x'(t), y'(t), z'(t))$. Applying Newton's law $m\mathbf{a} = \mathbf{F}$ componentwise, we get the system

$$\begin{aligned} mx'' &= F_1(t, x, y, z, x', y', z'), \\ my'' &= F_2(t, x, y, z, x', y', z'), \\ mz'' &= F_3(t, x, y, z, x', y', z') \end{aligned} \tag{2}$$

of three second-order equations with independent variable t and dependent variables x, y, z; the three right-hand-side functions F_1, F_2, F_3 are the components of the vector-valued function $\mathbf{F}$.

Initial Applications

Examples 1 through 3 further illustrate how systems of differential equations arise naturally in scientific problems.

Example 1

Consider the system of two masses and two springs shown in Fig. 4.1.1, with a given external force $f(t)$ acting on the right-hand mass m_2. We denote by $x(t)$ the displacement (to the right) of the mass m_1 from its static equilibrium position (when the system is motionless and in equilibrium and $f(t) = 0$) and by $y(t)$ the displacement of the mass m_2 from its static position. Thus the two springs are neither stretched nor compressed when x and y are zero.

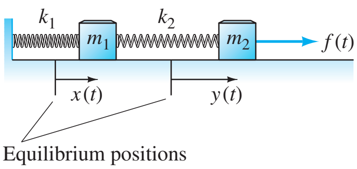

FIGURE 4.1.1. The mass-and-spring system of Example 1.

In the configuration in Fig. 4.1.1, the first spring is stretched x units and the second by $y - x$ units. We apply Newton's law of motion to the two "free body diagrams" shown in Fig. 4.1.2; we thereby obtain the system

$$\begin{aligned} m_1 x'' &= -k_1 x + k_2(y - x), \\ m_2 y'' &= -k_2(y - x) + f(t) \end{aligned} \tag{3}$$

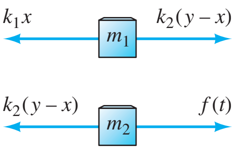

FIGURE 4.1.2. The "free body diagrams" for the system of Example 1.

of differential equations that the position functions $x(t)$ and $y(t)$ must satisfy. For instance, if $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$, and $f(t) = 40 \sin 3t$ in appropriate physical units, then the system in (3) reduces to

$$\begin{aligned} 2x'' &= -6x + 2y, \\ y'' &= 2x - 2y + 40 \sin 3t. \end{aligned} \tag{4}$$

■

Example 2

Consider two brine tanks connected as shown in Fig. 4.1.3. Tank 1 contains $x(t)$ pounds of salt in 100 gal of brine and tank 2 contains $y(t)$ pounds of salt in 200 gal of brine. The brine in each tank is kept uniform by stirring, and brine is pumped from each tank to the other at the rates indicated in Fig. 4.1.3. In addition, fresh water flows into tank 1 at 20 gal/min, and the brine in tank 2 flows out at 20 gal/min (so the total volume of brine in the two tanks remains constant). The salt concentrations in the two tanks are $x/100$ pounds per gallon and $y/200$ pounds per gallon, respectively. When we compute the rates of change of the amount of salt in the two tanks, we therefore get the system of differential equations that $x(t)$ and $y(t)$ must satisfy:

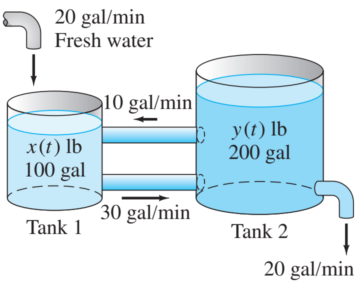

FIGURE 4.1.3. The two brine tanks of Example 2.

$$\begin{aligned} x' &= -30 \cdot \frac{x}{100} + 10 \cdot \frac{y}{200} = -\frac{3}{10}x + \frac{1}{20}y, \\ y' &= 30 \cdot \frac{x}{100} - 10 \cdot \frac{y}{200} - 20 \cdot \frac{y}{200} = \frac{3}{10}x - \frac{3}{20}y; \end{aligned}$$

that is,

$$\begin{aligned} 20x' &= -6x + y, \\ 20y' &= 6x - 3y. \end{aligned} \tag{5}$$

■

Example 3

Consider the electrical network shown in Fig. 4.1.4, where $I_1(t)$ denotes the current in the indicated direction through the inductor L and $I_2(t)$ denotes the current through the resistor R_2. The current through the resistor R_1 is $I = I_1 - I_2$ in the direction indicated. We recall Kirchhoff's voltage law to the effect that the (algebraic) sum of the voltage drops around any closed loop of such a network is zero. As in Section 3.7, the voltage drops across the three types of circuit elements are those shown in Fig. 4.1.5. We apply Kirchhoff's law to the left-hand loop of the network to obtain

$$2\frac{dI_1}{dt} + 50(I_1 - I_2) - 100 = 0, \tag{6}$$

because the voltage drop from the negative to the positive pole of the battery is -100. The right-hand loop yields the equation

$$125Q_2 + 25I_2 + 50(I_2 - I_1) = 0, \tag{7}$$

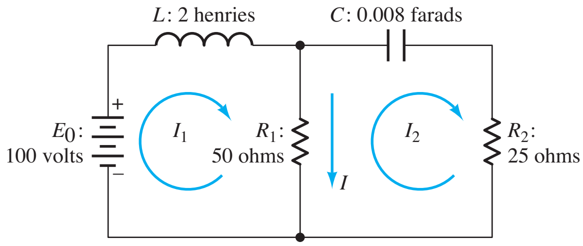

FIGURE 4.1.4. The electrical network of Example 3.

Circuit Element	Voltage Drop
Inductor	$L\dfrac{dI}{dt}$
Resistor	RI
Capacitor	$\dfrac{1}{C}Q$

FIGURE 4.1.5. Voltage drops across common circuit elements.

where $Q_2(t)$ is the charge on the capacitor. Because $dQ_2/dt = I_2$, differentiation of each side of Eq. (7) yields

$$125I_2 + 75\frac{dI_2}{dt} - 50\frac{dI_1}{dt} = 0. \tag{8}$$

After dividing Eqs. (6) and (8) by the factors 2 and −25, respectively, we get the system

$$\begin{aligned} \frac{dI_1}{dt} + 25I_1 - 25I_2 &= 50, \\ 2\frac{dI_1}{dt} - 3\frac{dI_2}{dt} - 5I_2 &= 0 \end{aligned} \tag{9}$$

of differential equations that the currents $I_1(t)$ and $I_2(t)$ must satisfy. ■

First-Order Systems

Consider a system of differential equations that can be solved for the highest-order derivatives of the dependent variables that appear, as explicit functions of t and lower-order derivatives of the dependent variables. For instance, in the case of a system of two second-order equations, our assumption is that it can be written in the form

$$\begin{aligned} x_1'' &= f_1(t, x_1, x_2, x_1', x_2'), \\ x_2'' &= f_2(t, x_1, x_2, x_1', x_2'). \end{aligned} \tag{10}$$

It is of both practical and theoretical importance that any such higher-order system can be transformed into an equivalent system of *first-order* equations.

To describe how such a transformation is accomplished, we consider first the "system" consisting of the single nth-order equation

➤ $$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}). \tag{11}$$

We introduce the dependent variables $x_1, x_2, \dots, x_n$ defined as follows:

$$x_1 = x, \quad x_2 = x', \quad x_3 = x'', \quad \dots, \quad x_n = x^{(n-1)}. \tag{12}$$

Note that $x_1' = x' = x_2$, $x_2' = x'' = x_3$, and so on. Hence the substitution of (12) in Eq. (11) yields the system

➤ $$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\vdots \\ x_{n-1}' &= x_n, \\ x_n' &= f(t, x_1, x_2, \dots, x_n) \end{aligned} \tag{13}$$

of n *first-order* equations. Evidently, this system is equivalent to the original nth-order equation in (11), in the sense that $x(t)$ is a solution of Eq. (11) if and only if the functions $x_1(t), x_2(t), \dots, x_n(t)$ defined in (12) satisfy the system of equations in (13).

Example 4 The third-order equation

$$x^{(3)} + 3x'' + 2x' - 5x = \sin 2t$$

is of the form in (11) with

$$f(t, x, x', x'') = 5x - 2x' - 3x'' + \sin 2t.$$

Hence the substitutions

$$x_1 = x, \quad x_2 = x' = x_1', \quad x_3 = x'' = x_2'$$

yield the system

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ x_3' &= 5x_1 - 2x_2 - 3x_3 + \sin 2t \end{aligned}$$

of three first-order equations. ■

It may appear that the first-order system obtained in Example 4 offers little advantage because we could use the methods of Chapter 3 to solve the original (linear) third-order equation. But suppose that we were confronted with the nonlinear equation

$$x'' = x^3 + (x')^3,$$

to which none of our earlier methods can be applied. The corresponding first-order system is

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= (x_1)^3 + (x_2)^3, \end{aligned} \tag{14}$$

and we will see in Section 4.3 that there exist effective numerical techniques for approximating the solution of essentially any first-order system. So in this case the transformation to a first-order system *is* advantageous. From a practical viewpoint, large systems of higher-order differential equations typically are solved numerically with the aid of the computer, and the first step is to transform such a system into a first-order system for which a standard computer program is available.

Example 5 The system

$$\begin{aligned} 2x'' &= -6x + 2y, \\ y'' &= 2x - 2y + 40\sin 3t \end{aligned} \tag{4}$$

of second-order equations was derived in Example 1. Transform this system into an equivalent first-order system.

Solution Motivated by the equations in (12), we define

$$x_1 = x, \quad x_2 = x' = x_1', \quad y_1 = y, \quad y_2 = y' = y_1'.$$

Then the system in (4) yields the system

$$\begin{aligned} x_1' &= x_2, \\ 2x_2' &= -6x_1 + 2y_1, \\ y_1' &= y_2, \\ y_2' &= 2x_1 - 2y_1 + 40\sin 3t \end{aligned} \tag{15}$$

of four first-order equations in the dependent variables x_1, x_2, y_1, and y_2. ■