GLOBAL EDITION

Calculus

for the Life Sciences

SECOND EDITION

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PEARSON

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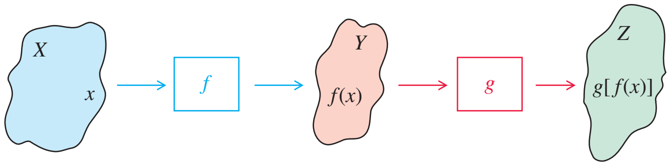

FIGURE 6

FOR REVIEW

You may want to review how to find the domain of a function. Domain was discussed in Section 1.3 on Properties of Functions.

Composite Function

Let f and g be functions. The **composite function**, or **composition**, of g and f is the function whose values are given by $g[f(x)]$ for all x in the domain of f such that $f(x)$ is in the domain of g. (Read $g[f(x)]$ as "g of f of x".)

EXAMPLE 1 Composite Functions

Let $f(x) = 2x - 1$ and $g(x) = \sqrt{3x + 5}$. Find the following.

(a) $g[f(4)]$

SOLUTION Find $f(4)$ first.

$$f(4) = 2 \cdot 4 - 1 = 8 - 1 = 7$$

Then

$$g[f(4)] = g[7] = \sqrt{3 \cdot 7 + 5} = \sqrt{26}.$$

(b) $f[g(4)]$

SOLUTION Since $g(4) = \sqrt{3 \cdot 4 + 5} = \sqrt{17}$,

$$f[g(4)] = 2 \cdot \sqrt{17} - 1 = 2\sqrt{17} - 1.$$

(c) $f[g(-2)]$

SOLUTION $f[g(-2)]$ does not exist, since -2 is not in the domain of g.

YOUR TURN 1 For the functions in Example 1, find $f[g(0)]$ and $g[f(0)]$.

TRY YOUR TURN 1

EXAMPLE 2 Composition of Functions

Let $f(x) = 2x^2 + 5x$ and $g(x) = 4x + 1$. Find the following.

(a) $f[g(x)]$

SOLUTION Using the given functions, we have

$$\begin{aligned} f[g(x)] &= f[4x + 1] \\ &= 2(4x + 1)^2 + 5(4x + 1) \\ &= 2(16x^2 + 8x + 1) + 20x + 5 \\ &= 32x^2 + 16x + 2 + 20x + 5 \\ &= 32x^2 + 36x + 7. \end{aligned}$$

(b) $g[f(x)]$

SOLUTION By the definition above, with f and g interchanged,

$$\begin{aligned} g[f(x)] &= g[2x^2 + 5x] \\ &= 4(2x^2 + 5x) + 1 \\ &= 8x^2 + 20x + 1. \end{aligned}$$

TRY YOUR TURN 2

YOUR TURN 2 Let $f(x) = 2x - 3$ and $g(x) = x^2 + 1$. Find $g[f(x)]$.

As Example 2 shows, it is not always true that $f[g(x)] = g[f(x)]$. In fact, it is rare to find two functions f and g such that $f[g(x)] = g[f(x)]$. The domain of both composite functions given in Example 2 is the set of all real numbers.

EXAMPLE 3 Composition of Functions

Write each function as the composition of two functions f and g so that $h(x) = f[g(x)]$.

(a) $h(x) = 2(4x + 1)^2 + 5(4x + 1)$

SOLUTION Let $f(x) = 2x^2 + 5x$ and $g(x) = 4x + 1$. Then $f[g(x)] = f(4x + 1) = 2(4x + 1)^2 + 5(4x + 1)$. Notice that $h(x)$ here is the same as $f[g(x)]$ in Example 2(a).

(b) $h(x) = \sqrt{1 - x^2}$

SOLUTION One way to do this is to let $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$. Another choice is to let $f(x) = \sqrt{1 - x}$ and $g(x) = x^2$. Verify that with either choice, $f[g(x)] = \sqrt{1 - x^2}$. For the purposes of this section, the first choice is better; it is useful to think of f as being the function on the outer layer and g as the function on the inner layer. With this function h, we see a square root on the outer layer, and when we peel that away we see $1 - x^2$ on the inside.

TRY YOUR TURN 3

YOUR TURN 3 Write $h(x) = (2x - 3)^3$ as a composition of two functions f and g so that $h(x) = f[g(x)]$.

The Chain Rule Suppose $f(x) = x^2$ and $g(x) = 5x^3 + 2$. What is the derivative of $h(x) = f[g(x)] = (5x^3 + 2)^2$? At first you might think the answer is just $h'(x) = 2(5x^3 + 2) = 10x^3 + 4$ by using the power rule. You can check this answer by multiplying out $h(x) = (5x^3 + 2)^2 = 25x^6 + 20x^3 + 4$. Now calculate $h'(x) = 150x^5 + 60x^2$. The guess using the power rule was clearly wrong! The error is that the power rule applies to x raised to a power, not to some other function of x raised to a power.

How, then, could we take the derivative of $p(x) = (5x^3 + 2)^{20}$? This seems far too difficult to multiply out. Fortunately, there is a way. Notice from the previous paragraph that $h'(x) = 150x^5 + 60x^2 = 2(5x^3 + 2)15x^2$. So the original guess was almost correct, except it was missing the factor of $15x^2$, which just happens to be $g'(x)$. This is not a coincidence. To see why the derivative of $f[g(x)]$ involves taking the derivative of f and then multiplying by the derivative of g, let us consider a realistic example, the question from the beginning of this section.

EXAMPLE 4 Area of an Oil Slick

A leaking oil well off the Gulf Coast is spreading a circular film of oil over the water surface. At any time t (in minutes) after the beginning of the leak, the radius of the circular oil slick (in feet) is given by

$$r(t) = 4t.$$

Find the rate of change of the area of the oil slick with respect to time.

APPLY IT

SOLUTION We first find the rate of change in the radius over time by finding dr/dt:

$$\frac{dr}{dt} = 4.$$

This value indicates that the radius is increasing by 4 ft each minute.

The area of the oil slick is given by

$$A(r) = \pi r^2, \quad \text{with} \quad \frac{dA}{dr} = 2\pi r.$$

The derivative, dA/dr, gives the rate of change in area per unit increase in the radius.

As these derivatives show, the radius is increasing at a rate of 4 ft/min, and for each foot that the radius increases, the area increases by $2\pi r$ ft^2. It seems reasonable, then, that the area is increasing at a rate of

$$2\pi r \text{ ft}^2/\text{ft} \times 4 \text{ ft/min} = 8\pi r \text{ ft}^2/\text{min}.$$

That is,

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot 4 = 8\pi r.$$

Notice that because area (A) is a function of radius (r), which is a function of time (t), area as a function of time is a composition of two functions, written $A(r(t))$. The last step, then, can also be written as

$$\frac{dA}{dt} = \frac{d}{dt}A[r(t)] = A'[r(t)] \cdot r'(t) = 2\pi r \cdot 4 = 8\pi r.$$

Finally, we can substitute $r(t) = 4t$ to get the derivative in terms of t:

$$\frac{dA}{dt} = 8\pi r = 8\pi(4t) = 32\pi t.$$

The rate of change of the area of the oil slick with respect to time is $32\pi t$ ft^2/min.

To check the result of Example 4, use the fact that $r = 4t$ and $A = \pi r^2$ to get the same result:

$$A = \pi(4t)^2 = 16\pi t^2, \qquad \text{with} \qquad \frac{dA}{dt} = 32\pi t.$$

The product used in Example 4,

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt},$$

is an example of the **chain rule**, which is used to find the derivative of a composite function.

Chain Rule

If y is a function of u, say $y = f(u)$, and if u is a function of x, say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

One way to remember the chain rule is to pretend that dy/du and du/dx are fractions, with du "canceling out." The proof of the chain rule requires advanced concepts and, therefore, is not given here.

EXAMPLE 5 Chain Rule

Find dy/dx if $y = (3x^2 - 5x)^{1/2}$.

SOLUTION Let $y = u^{1/2}$, and $u = 3x^2 - 5x$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2}u^{-1/2} \cdot (6x - 5).$$

Replacing u with $3x^2 - 5x$ gives

YOUR TURN 4 Find dy/dx if $y = (5x^2 - 6x)^{-2}$.

$$\frac{dy}{dx} = \frac{1}{2}(3x^2 - 5x)^{-1/2}(6x - 5) = \frac{6x - 5}{2(3x^2 - 5x)^{1/2}}.$$

TRY YOUR TURN 4

The following alternative version of the chain rule is stated in terms of composite functions.

Chain Rule (Alternative Form)

If $y = f[g(x)]$, then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

(To find the derivative of $f[g(x)]$, find the derivative of $f(x)$, replace each x with $g(x)$, and then multiply the result by the derivative of $g(x)$.)

In words, the chain rule tells us to first take the derivative of the outer function, then multiply it by the derivative of the inner function.

EXAMPLE 6 Chain Rule

Use the chain rule to find $D_x(x^2 + 5x)^8$.

SOLUTION As in Example 3(b), think of this as a function with layers. The outer layer is something being raised to the 8th power, so let $f(x) = x^8$. Once this layer is peeled away, we see that the inner layer is $x^2 + 5x$, so $g(x) = x^2 + 5x$. Then $(x^2 + 5x)^8 = f[g(x)]$ and

$$D_x(x^2 + 5x)^8 = f'[g(x)]g'(x).$$

Here $f'(x) = 8x^7$, with $f'[g(x)] = 8[g(x)]^7 = 8(x^2 + 5x)^7$ and $g'(x) = 2x + 5$.

$$\begin{aligned} D_x(x^2 + 5x)^8 &= f'[g(x)]g'(x) \\ &= 8[g(x)]^7 g'(x) \\ &= 8(x^2 + 5x)^7(2x + 5) \end{aligned}$$

YOUR TURN 5 Find $D_x(x^2 - 7)^{10}$.

TRY YOUR TURN 5

CAUTION

(a) A common error is to forget to multiply by $g'(x)$ when using the chain rule. Remember, the derivative must involve a "chain," or product, of derivatives.

(b) Another common mistake is to write the derivative as $f'[g'(x)]$. Remember to leave $g(x)$ unchanged in $f'[g(x)]$ and then to multiply by $g'(x)$.

One way to avoid both of the errors described above is to remember that the chain rule is a two-step process. In Example 6, the first step was taking the derivative of the power, and the second step was multiplying by $g'(x)$. Forgetting to multiply by $g'(x)$ would be an erroneous one-step process. The other erroneous one-step process is to take the derivative inside the power, getting $f'[g'(x)]$, or $8(2x + 5)^7$ in Example 6.

Sometimes both the chain rule and either the product or quotient rule are needed to find a derivative, as the next examples show.

EXAMPLE 7 Derivative Rules

Find the derivative of $y = 4x(3x + 5)^5$.

SOLUTION Write $4x(3x + 5)^5$ as the product

$$(4x) \cdot (3x + 5)^5.$$

To find the derivative of $(3x + 5)^5$, let $g(x) = 3x + 5$, with $g'(x) = 3$. Now use the product rule and the chain rule.

$$\frac{dy}{dx} = 4x\underbrace{[5(3x + 5)^4 \cdot 3]}_{\text{Derivative of } (3x+5)^5} + (3x + 5)^5\underbrace{(4)}_{\text{Derivative of } 4x}$$

$$= 60x(3x + 5)^4 + 4(3x + 5)^5$$

$$= 4(3x + 5)^4[15x + (3x + 5)^1] \quad \text{Factor out the greatest common factor, } 4(3x + 5)^4.$$

$$= 4(3x + 5)^4(18x + 5) \quad \text{Simplify inside brackets.}$$

TRY YOUR TURN 6

YOUR TURN 6 Find the derivative of $y = x^2(5x - 1)^3$.

EXAMPLE 8 Derivative Rules

Find $D_x\left[\frac{(3x + 2)^7}{x - 1}\right]$.

SOLUTION Use the quotient rule and the chain rule.

$$D_x\left[\frac{(3x + 2)^7}{x - 1}\right] = \frac{(x - 1)[7(3x + 2)^6 \cdot 3] - (3x + 2)^7(1)}{(x - 1)^2}$$

$$= \frac{21(x - 1)(3x + 2)^6 - (3x + 2)^7}{(x - 1)^2}$$

$$= \frac{(3x + 2)^6[21(x - 1) - (3x + 2)]}{(x - 1)^2} \quad \text{Factor out the greatest common factor, } (3x + 2)^6.$$

$$= \frac{(3x + 2)^6[21x - 21 - 3x - 2]}{(x - 1)^2} \quad \text{Distribute.}$$

$$= \frac{(3x + 2)^6(18x - 23)}{(x - 1)^2} \quad \text{Simplify inside brackets.}$$

TRY YOUR TURN 7

YOUR TURN 7

Find $D_x\left[\frac{(4x - 1)^3}{x + 3}\right]$.

Some applications requiring the use of the chain rule are illustrated in the next two examples.

EXAMPLE 9 Tasmanian Devil

Named for the only place where it is currently known to live, the extremely voracious Tasmanian Devil often preys on animals larger than itself. Researchers have identified the mathematical relationship

$$L(w) = 2.265w^{2.543},$$

where w (in kg) is the weight and $L(w)$ (in mm) is the length of the Tasmanian Devil. ***Source: Journal of Mammalogy.*** Suppose that the weight of a particular Tasmanian Devil can be estimated by the function

$$w(t) = 0.125 + 0.18t,$$

where $w(t)$ is the weight (in kg) and t is the age, in weeks, of a Tasmanian Devil that is less than one year old. How fast is the length of a 30-week-old Tasmanian Devil changing?

SOLUTION We want to find dL/dt, the rate of change of the length of a Tasmanian Devil. By the chain rule,

$$\frac{dL}{dt} = \frac{dL}{dw} \cdot \frac{dw}{dt}.$$

First find dL/dw, as follows.

$$\frac{dL}{dw} = (2.265)(2.543)w^{2.543-1} \approx 5.76w^{1.543}$$

Also,

$$\frac{dw}{dt} = 0.18.$$

Therefore,

$$\frac{dL}{dt} = 5.76w^{1.543} \cdot 0.18.$$

Since the Tasmanian Devil is 30 weeks old, $t = 30$ and

$$w(30) = 0.125 + 0.18(30) = 5.525 \text{ kg}.$$

Putting this all together, we have

$$\frac{dL}{dt} \approx 5.76(5.525)^{1.543}(0.18) \approx 14.5.$$

The length of a 30-week-old Tasmanian Devil is increasing at the rate of about 14.5 mm per week.

Alternatively, we can find dL/dt by first substituting the formula for $w(t)$ into $L(w)$. Then directly take the derivative of $L(t)$ with respect to t.

EXAMPLE 10 Compound Interest

Suppose a sum of \$500 is deposited in an account with an interest rate of r percent per year compounded monthly. At the end of 10 years, the balance in the account (as illustrated in Figure 7) is given by

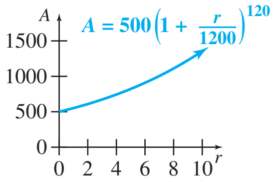

FIGURE 7

$$A = 500\left(1 + \frac{r}{1200}\right)^{120}.$$

Find the rate of change of A with respect to r if $r = 5$ or 7.*

SOLUTION First find dA/dr using the chain rule.

$$\frac{dA}{dr} = (120)(500)\left(1 + \frac{r}{1200}\right)^{119}\left(\frac{1}{1200}\right)$$
$$= 50\left(1 + \frac{r}{1200}\right)^{119}$$

If $r = 5$,

$$\frac{dA}{dr} = 50\left(1 + \frac{5}{1200}\right)^{119}$$
$$\approx 82.01,$$

or \$82.01 per percentage point. If $r = 7$,

$$\frac{dA}{dr} = 50\left(1 + \frac{7}{1200}\right)^{119}$$
$$\approx 99.90,$$

or \$99.90 per percentage point.

*Notice that r is given here as an integer percent, rather than as a decimal, which is why the formula for compound interest has 1200 where you would expect to see 12. This leads to a simpler interpretation of the derivative.