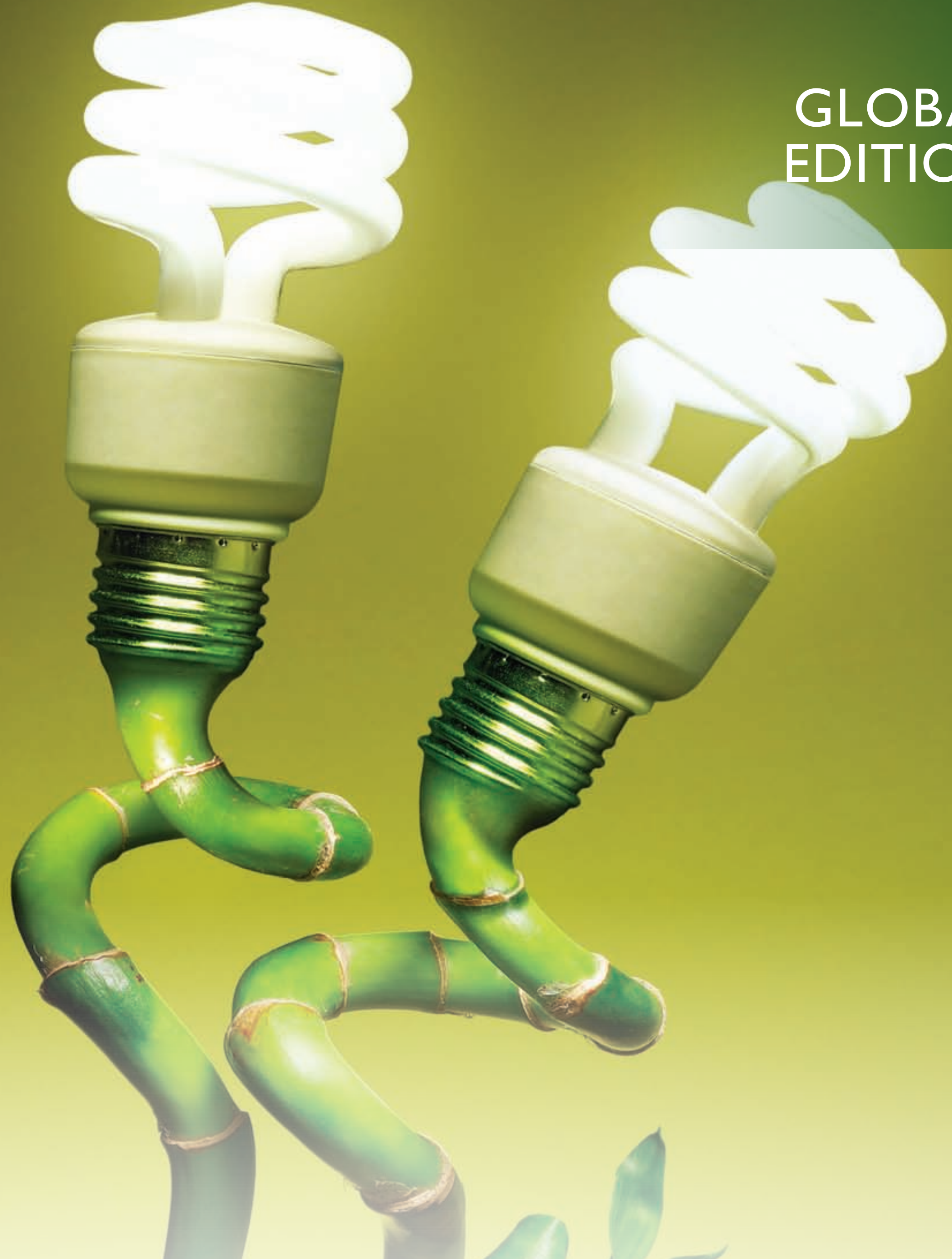

GLOBAL EDITION

College Mathematics

for Business, Economics, Life Sciences, and Social Sciences

THIRTEENTH EDITION

Raymond A. Barnett • Michael R. Ziegler • Karl E. Byleen

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PEARSON

COLLEGE MATHEMATICS

FOR BUSINESS, ECONOMICS, LIFE SCIENCES, AND SOCIAL SCIENCES

Thirteenth Edition

Global Edition

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There will be $(m + 2)(m + 1)/2 = 10$ rows in the table of basic solutions. Because there are two decision variables, x_1 and x_2, we assign two zeros to each row of the table in all possible combinations.

x_1	x_2	s_1	s_2	s_3
0	0			
0		0		
0			0	
0				0
	0	0		
	0		0	
	0			0
		0	0	
		0		0
			0	0

We complete the table working one row at a time. We substitute 0's for the two variables indicated by the row, in the *e*-system (5). The result is a system of three equations in three variables, which can be solved by Gauss–Jordan elimination, or by another of our standard methods. Table 5 shows all ten basic solutions, and the values of the objective function, $P = 40x_1 + 50x_2$, at the five basic feasible solutions.

Table 5 **The Table Method**

x_1	x_2	s_1	s_2	s_3	$P = 40x_1 + 50x_2$
0	0	72	45	72	0
0	12	0	9	36	600
0	15	−18	0	27	–
0	24	−72	−27	0	–
72	0	0	−27	−72	–
45	0	27	0	−18	–
36	0	36	9	0	1,440
18	9	0	0	9	1,170
24	8	0	−3	0	–
27	6	9	0	0	1,380

We conclude that

$$\text{Max } P = 1{,}440 \text{ at } x_1 = 36, x_2 = 0$$

Matched Problem 3 Construct the table of basic solutions and use it to solve the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & P = 36x_1 + 24x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 8 \\ & x_1 + x_2 \le 5 \\ & 2x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Basic and Nonbasic Variables

The basic solutions associated with a linear programming problem are found by assigning the value 0 to certain decision variables (the x_i's) and slack variables (the s_i's). Consider, for example, row 2 of Table 5. That row shows the basic solution $(x_1, x_2, s_1, s_2, s_3) = (0, 12, 0, 9, 36)$. It is customary to refer to the variables that are assigned the value 0 as **nonbasic variables**, and to the others as **basic variables**. So for the basic solution of row 2, the basic variables are x_2, s_2, and s_3; the nonbasic variables are x_1 and s_1.

Note that the classification of variables as basic or nonbasic depends on the basic solution. Row 8 of Table 5 shows the basic solution $(x_1, x_2, s_1, s_2, s_3) = (18, 9, 0, 0, 9)$. For row 8, the basic variables are x_1, x_2, and s_3; the nonbasic variables are s_1 and s_2.

EXAMPLE 4 Basic and Nonbasic Variables Refer to Table 5. For the basic solution $(x_1, x_2, s_1, s_2, s_3) = (36, 0, 36, 9, 0)$ in row 7 of Table 5, classify the variables as basic or nonbasic.

SOLUTION The basic variables are x_1, s_1, and s_2. The other variables, x_2 and s_3, were assigned the value 0, and therefore are nonbasic.

Matched Problem 4 Refer to Table 5. For the basic solution $(x_1, x_2, s_1, s_2, s_3) = (27, 6, 9, 0, 0)$ of row 10, classify the variables as basic or nonbasic.

Explore and Discuss 2 Use the table method to solve the following linear programming problem, and explain why one of the rows in the table cannot be completed to a basic solution:

$$\begin{aligned} \text{Maximize} \quad & P = 10x_1 + 12x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2 \\ & x_1 + x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Summary

The examples in this section illustrate the table method when there are two decision variables. But the method can be used when there are k decision variables, where k is any positive integer.

The number of ways in which r objects can be chosen from a set of n objects, without regard to order, is denoted by ${}_nC_r$ and given by the formula

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

(The formula, giving the number of combinations of n distinct objects taken r at a time, is explained and derived in Chapter 7). If there are k decision variables and m problem constraints in a linear programming problem, then the number of rows in the table of basic solutions is ${}_{k+m}C_k$, because this is the number of ways of selecting k of the $k + m$ variables to be assigned the value 0.

PROCEDURE The Table Method (k Decision Variables)

Assume that a standard maximization problem in standard form has k decision variables $x_1, x_2, \ldots, x_k$, and m problem constraints.

Step 1 Use slack variables $s_1, s_2, \ldots, s_m$ to convert the i-system to an e-system.

Step 2 Form a table with ${}_{k+m}C_k$ rows and $k + m$ columns labeled $x_1, x_2, \ldots, x_k, s_1, s_2, \ldots, s_m$. In the first row, assign 0 to $x_1, x_2, \ldots, x_k$. Continue until the rows contain all possible combinations of assigning k 0's to the variables.

(Continued)

Step 3 Complete each row to a solution of the *e*-system, if possible. Because k of the variables have the value 0, this involves solving a system of m linear equations in m variables. Use the Gauss–Jordan method, or another method if you find it easier. If the system has no solutions, or infinitely many solutions, do not complete the row.

Step 4 Solve the linear programming problem by finding the maximum value of P over those completed rows that have no negative values (that is, over the basic feasible solutions).

The benefit of the table method is that it gives a procedure for **finding all corner points of the feasible region without drawing a graph**.

Unfortunately, the number of rows in the table becomes too large to be practical, even for computers, when the number of decision variables and problem constraints is large. For example, with $k = 30$ decision variables and $m = 35$ problem constraints, the number of rows is

$$_{65}C_{30} \approx 3 \times 10^{18}$$

We need a procedure for finding the optimal solution of a linear programming problem without having to find every corner point. The *simplex method*, discussed in the next section, is such a procedure. It gives a practical method for solving large linear programming problems.

Exercises 6.1

Skills Warm-up Exercises

W *In Problems 1–4, if necessary, review Section B.3.*

1. In how many ways can two variables be chosen from x_1, x_2, s_1, s_2, s_3 and assigned the value 0?

2. In how many ways can two variables be chosen from x_1, x_2, s_1, s_2 and assigned the value 0?

3. In how many ways can two variables be chosen from $x_1, x_2, x_3, s_1, s_2, s_3$ and assigned the value 0?

4. In how many ways can three variables be chosen from $x_1, x_2, x_3, s_1, s_2, s_3, s_4, s_5$ and assigned the value 0?

Problems 5–8 refer to the system

$$\begin{aligned} 2x_1 + 5x_2 + s_1 &= 10 \\ x_1 + 3x_2 + s_2 &= 8 \end{aligned}$$

5. Find the solution of the system for which $x_1 = 0, s_1 = 0$.

6. Find the solution of the system for which $x_1 = 0, s_2 = 0$.

7. Find the solution of the system for which $x_2 = 0, s_2 = 0$.

8. Find the solution of the system for which $x_2 = 0, s_1 = 0$.

In Problems 9–16, write the e-system obtained via slack variables for the given linear programming problem.

9. Maximize $P = 5x_1 + 7x_2$
subject to
$$\begin{aligned} 2x_1 + 3x_2 &\le 9 \\ 6x_1 + 7x_2 &\le 13 \\ x_1, x_2 &\ge 0 \end{aligned}$$

10. Maximize $P = 35x_1 + 25x_2$
subject to
$$\begin{aligned} 10x_1 + 15x_2 &\le 100 \\ 5x_1 + 20x_2 &\le 120 \\ x_1, x_2 &\ge 0 \end{aligned}$$

11. Maximize $P = 3x_1 + 5x_2$
subject to
$$\begin{aligned} 12x_1 - 14x_2 &\le 55 \\ 19x_1 + 5x_2 &\le 40 \\ -8x_1 + 11x_2 &\le 64 \\ x_1, x_2 &\ge 0 \end{aligned}$$

12. Maximize $P = 13x_1 + 25x_2$
subject to
$$\begin{aligned} 3x_1 + 5x_2 &\le 27 \\ 8x_1 + 3x_2 &\le 19 \\ 4x_1 + 9x_2 &\le 34 \\ x_1, x_2 &\ge 0 \end{aligned}$$

13. Maximize $P = 4x_1 + 7x_2$
subject to $6x_1 + 5x_2 \le 18$
$x_1, x_2 \ge 0$

14. Maximize $P = 13x_1 + 8x_2$
subject to $x_1 + 2x_2 \le 20$
$x_1, x_2 \ge 0$

15. Maximize $P = x_1 + 2x_2$
subject to $4x_1 - 3x_2 \le 12$
$5x_1 + 2x_2 \le 25$
$-3x_1 + 7x_2 \le 32$
$2x_1 + x_2 \le 9$
$x_1, x_2 \ge 0$

16. Maximize $P = 8x_1 + 9x_2$
subject to $30x_1 - 25x_2 \le 75$
$10x_1 + 13x_2 \le 30$
$5x_1 + 18x_2 \le 40$
$40x_1 + 36x_2 \le 85$
$x_1, x_2 \ge 0$

Problems 17–26 refer to the table below of the six basic solutions to the e-system

$$\begin{aligned} 2x_1 + 3x_2 + s_1 \quad &= 24 \\ 4x_1 + 3x_2 \quad + s_2 &= 36 \end{aligned}$$

	x_1	x_2	s_1	s_2
(A)	0	0	24	36
(B)	0	8	0	12
(C)	0	12	−12	0
(D)	12	0	0	−12
(E)	9	0	6	0
(F)	6	4	0	0

17. In basic solution (A), which variables are basic?

18. In basic solution (B), which variables are nonbasic?

19. In basic solution (C), which variables are nonbasic?

20. In basic solution (D), which variables are basic?

21. In basic solution (E), which variables are nonbasic?

22. In basic solution (F), which variables are basic?

23. Which of the six basic solutions are feasible? Explain.

24. Which of the basic solutions are not feasible? Explain.

25. Use the basic feasible solutions to maximize $P = 2x_1 + 5x_2$.

26. Use the basic feasible solutions to maximize $P = 8x_1 + 5x_2$.

Problems 27–36 refer to the partially completed table below of the 10 basic solutions to the e-system

$$\begin{aligned} x_1 + x_2 + s_1 \qquad\quad &= 24 \\ 2x_1 + x_2 \quad + s_2 \quad &= 30 \\ 4x_1 + x_2 \qquad + s_3 &= 48 \end{aligned}$$

	x_1	x_2	s_1	s_2	s_3
(A)	0	0	24	30	48
(B)	0	24	0	6	24
(C)	0	30	−6	0	18
(D)	0	48	−24	−18	0
(E)	24	0	0	−18	−48
(F)	15	0	9	0	−12
(G)		0			0
(H)			0	0	
(I)			0		0
(J)				0	0

27. In basic solution (C), which variables are basic?

28. In basic solution (E), which variables are nonbasic?

29. In basic solution (G), which variables are nonbasic?

30. In basic solution (I), which variables are basic?

31. Which of the basic solutions (A) through (F) are not feasible? Explain.

32. Which of the basic solutions (A) through (F) are feasible? Explain.

33. Find basic solution (G).

34. Find basic solution (H).

35. Find basic solution (I).

36. Find basic solution (J).

In Problems 37–44, convert the given i-system to an e-system using slack variables. Then construct a table of all basic solutions of the e-system. For each basic solution, indicate whether or not it is feasible.

37. $4x_1 + 5x_2 \le 20$
$x_1, x_2 \ge 0$

38. $3x_1 + 8x_2 \le 24$
$x_1, x_2 \ge 0$

39. $x_1 + x_2 \le 6$
$x_1 + 4x_2 \le 12$
$x_1, x_2 \ge 0$

40. $5x_1 + x_2 \le 15$
$x_1 + x_2 \le 7$
$x_1, x_2 \ge 0$

41. $2x_1 + 5x_2 \le 20$
$x_1 + 2x_2 \le 9$
$x_1, x_2 \ge 0$

42. $x_1 + 3x_2 \le 18$
$5x_1 + 4x_2 \le 35$
$x_1, x_2 \ge 0$

43. $x_1 + 2x_2 \le 24$
$x_1 + x_2 \le 15$
$2x_1 + x_2 \le 24$
$x_1, x_2 \ge 0$

44. $5x_1 + 4x_2 \le 240$
$5x_1 + 2x_2 \le 150$
$5x_1 + x_2 \le 120$
$x_1, x_2 \ge 0$

In Problems 45–50, graph the system of inequalities from the given problem, and list the corner points of the feasible region. Verify that the corner points of the feasible region correspond to the basic feasible solutions of the associated e-system.

45. Problem 37

46. Problem 38

47. Problem 39

48. Problem 40

49. Problem 41

50. Problem 42

In Problems 51–58, solve the given linear programming problem using the table method (the table of basic solutions was constructed in Problems 37–44).

51. Maximize $P = 10x_1 + 9x_2$
subject to $4x_1 + 5x_2 \le 20$
$x_1, x_2 \ge 0$

52. Maximize $P = 4x_1 + 7x_2$
subject to $3x_1 + 8x_2 \le 24$
$x_1, x_2 \ge 0$

53. Maximize $P = 15x_1 + 20x_2$
subject to $x_1 + x_2 \le 6$
$x_1 + 4x_2 \le 12$
$x_1, x_2 \ge 0$

54. Maximize $P = 5x_1 + 20x_2$
subject to $5x_1 + x_2 \le 15$
$x_1 + x_2 \le 7$
$x_1, x_2 \ge 0$

55. Maximize $P = 25x_1 + 10x_2$
subject to $2x_1 + 5x_2 \le 20$
$x_1 + 2x_2 \le 9$
$x_1, x_2 \ge 0$

56. Maximize $P = 40x_1 + 50x_2$
subject to $x_1 + 3x_2 \le 18$
$5x_1 + 4x_2 \le 35$
$x_1, x_2 \ge 0$

57. Maximize $P = 30x_1 + 40x_2$
subject to $x_1 + 2x_2 \le 24$
$x_1 + x_2 \le 15$
$2x_1 + x_2 \le 24$
$x_1, x_2 \ge 0$

58. Maximize $P = x_1 + x_2$
subject to $5x_1 + 4x_2 \le 240$
$5x_1 + 2x_2 \le 150$
$5x_1 + x_2 \le 120$
$x_1, x_2 \ge 0$

59. A linear programming problem has four decision variables x_1, x_2, x_3, x_4, and six problem constraints. How many rows are there in the table of basic solutions of the associated *e*-system?

60. A linear programming problem has five decision variables x_1, x_2, x_3, x_4, x_5 and six problem constraints. How many rows are there in the table of basic solutions of the associated *e*-system?

61. A linear programming problem has 30 decision variables $x_1, x_2, \ldots, x_{30}$ and 42 problem constraints. How many rows are there in the table of basic solutions of the associated *e*-system? (Write the answer using scientific notation.)

62. A linear programming problem has 40 decision variables $x_1, x_2, \ldots, x_{40}$ and 85 problem constraints. How many rows are there in the table of basic solutions of the associated *e*-system? (Write the answer using scientific notation.)

Answers to Matched Problems

1. $(x_1, x_2, s_1, s_2) = (7, 0, 0, 13)$

2.

x_1	x_2	s_1	s_2	$P = 30x_1 + 40x_2$
0	0	24	36	0
0	8	0	12	320
0	12	−12	0	–
12	0	0	−12	–
9	0	6	0	270
6	4	0	0	340

Max $P = 340$ at $x_1 = 6, x_2 = 4$

3.

x_1	x_2	s_1	s_2	s_3	$P = 36x_1 + 24x_2$
0	0	8	5	8	0
0	4	0	1	4	96
0	5	−2	0	3	–
0	8	−8	−3	0	–
8	0	0	−3	−8	–
5	0	3	0	−2	–
4	0	4	1	0	144
2	3	0	0	1	144
8/3	8/3	0	−1/3	0	–
3	2	1	0	0	156

Max $P = 156$ at $x_1 = 3, x_2 = 2$

4. x_1, x_2, and s_1 are basic; s_2 and s_3 are nonbasic

6.2 The Simplex Method: Maximization with Problem Constraints of the Form ≤

- Initial System
- Simplex Tableau
- Pivot Operation
- Interpreting the Simplex Process Geometrically
- Simplex Method Summarized
- Application

Now we can develop the simplex method for a standard maximization problem. The simplex method is most useful when used with computers. Consequently, it is not intended that you become an expert in manually solving linear programming problems using the simplex method. But it is important that you become proficient in constructing the models for linear programming problems so that they can be solved using a computer, and it is also important that you develop skill in interpreting the results. One way to gain this proficiency and interpretive skill is to set up and manually solve a number of fairly simple linear programming problems using the simplex method. This is the main goal in this section and in Sections 6.3 and 6.4. To assist you in learning to develop the models, the answer sections for Exercises 6.2, 6.3, and 6.4 contain both the model and its solution.

Initial System

We will introduce the concepts and procedures involved in the simplex method through an example—the tent production example discussed earlier. We restate the problem here in standard form for convenient reference:

$$\begin{aligned} \text{Maximize } P &= 50x_1 + 80x_2 && \text{Objective function} \\ \text{subject to } \quad x_1 + 2x_2 &\le 32 && \text{Problem constraints} \\ 3x_1 + 4x_2 &\le 84 && \\ x_1, x_2 &\ge 0 && \text{Nonnegative constraints} \end{aligned} \qquad (1)$$

Introducing slack variables s_1 and s_2, we convert the problem constraint inequalities in problem (1) into the following system of problem constraint equations:

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 32 \\ 3x_1 + 4x_2 + s_2 &= 84 \\ x_1, x_2, s_1, s_2 &\ge 0 \end{aligned} \qquad (2)$$

Since a basic solution of system (2) is not feasible if it contains any negative values, we have also included the nonnegative constraints for both the decision variables x_1 and x_2 and the slack variables s_1 and s_2. From our discussion in Section 6.1, we know that out of the infinitely many solutions to system (2), an optimal solution is one of the basic feasible solutions, which correspond to the corner points of the feasible region.

As part of the simplex method we add the objective function equation $P = 50x_1 + 80x_2$ in the form $-50x_1 - 80x_2 + P = 0$ to system (2) to create what is called the **initial system**:

$$\begin{aligned} x_1 + 2x_2 + s_1 &= 32 \\ 3x_1 + 4x_2 + s_2 &= 84 \\ -50x_1 - 80x_2 + P &= 0 \\ x_1, x_2, s_1, s_2 &\ge 0 \end{aligned} \qquad (3)$$

When we add the objective function equation to system (2), we must slightly modify the earlier definitions of basic solution and basic feasible solution so that they apply to the initial system (3).

DEFINITION Basic Solutions and Basic Feasible Solutions for Initial Systems

1. The objective function variable P is always selected as a basic variable.
2. Note that a basic solution of system (3) is also a basic solution of system (2) after P is deleted.

(Continued)