GLOBAL EDITION

Calculus

Early Transcendentals

SECOND EDITION

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ALWAYS LEARNING

PEARSON

Calculus

EARLY TRANSCENDENTALS

SECTION 5.3 EXERCISES

Review Questions

1. Suppose A is an area function of f. What is the relationship between f and A?

2. Suppose F is an antiderivative of f and A is an area function of f. What is the relationship between F and A?

3. Explain in words and write mathematically how the Fundamental Theorem of Calculus is used to evaluate definite integrals.

4. Let $f(x) = c$, where c is a positive constant. Explain why an area function of f is an increasing function.

5. The linear function $f(x) = 3 - x$ is decreasing on the interval $[0, 3]$. Is the area function for f (with left endpoint 0) increasing or decreasing on the interval $[0, 3]$? Draw a picture and explain.

6. Evaluate $\int_0^2 3x^2\,dx$ and $\int_{-2}^2 3x^2\,dx$.

7. Explain in words and express mathematically the inverse relationship between differentiation and integration as given by Part 1 of the Fundamental Theorem of Calculus.

8. Why can the constant of integration be omitted from the antiderivative when evaluating a definite integral?

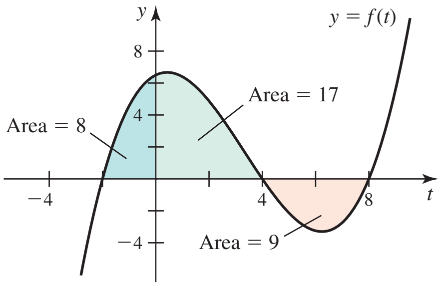
9. Evaluate $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt$ and $\dfrac{d}{dx}\displaystyle\int_a^b f(t)\,dt$, where a and b are constants.

10. Explain why $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Basic Skills

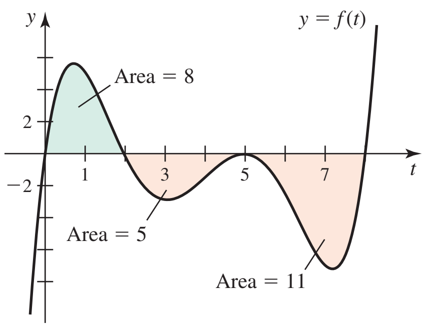
11. **Area functions** The graph of f is shown in the figure. Let $A(x) = \int_{-2}^x f(t)\,dt$ and $F(x) = \int_4^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

a. $A(-2)$ **b.** $F(8)$ **c.** $A(4)$ **d.** $F(4)$ **e.** $A(8)$



12. **Area functions** The graph of f is shown in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

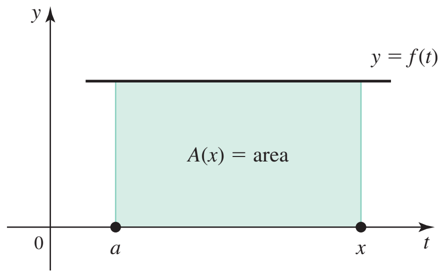
a. $A(2)$ **b.** $F(5)$ **c.** $A(0)$ **d.** $F(8)$
e. $A(8)$ **f.** $A(5)$ **g.** $F(2)$



13–16. Area functions for constant functions *Consider the following functions f and real numbers a (see figure).*

a. *Find and graph the area function $A(x) = \int_a^x f(t)\,dt$ for f.*

b. *Verify that $A'(x) = f(x)$.*



13. $f(t) = 5,\ a = 0$

14. $f(t) = 10,\ a = 4$

15. $f(t) = 5,\ a = -5$

16. $f(t) = 2,\ a = -3$

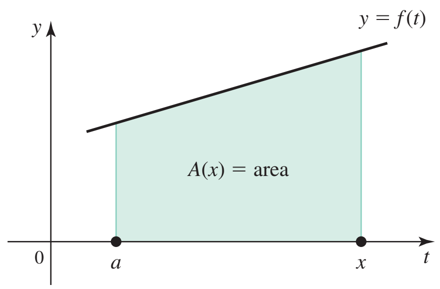
17. **Area functions for the same linear function** Let $f(t) = t$ and consider the two area functions $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$.
   a. Evaluate $A(2)$ and $A(4)$. Then use geometry to find an expression for $A(x)$, for $x \geq 0$.
   b. Evaluate $F(4)$ and $F(6)$. Then use geometry to find an expression for $F(x)$, for $x \geq 2$.
   c. Show that $A(x) - F(x)$ is a constant and that $A'(x) = F'(x) = f(x)$.

18. **Area functions for the same linear function** Let $f(t) = 2t - 2$ and consider the two area functions $A(x) = \int_1^x f(t)\,dt$ and $F(x) = \int_4^x f(t)\,dt$.
   a. Evaluate $A(2)$ and $A(3)$. Then use geometry to find an expression for $A(x)$, for $x \geq 1$.
   b. Evaluate $F(5)$ and $F(6)$. Then use geometry to find an expression for $F(x)$, for $x \geq 4$.
   c. Show that $A(x) - F(x)$ is a constant and that $A'(x) = F'(x) = f(x)$.

19–22. Area functions for linear functions *Consider the following functions f and real numbers a (see figure).*

a. *Find and graph the area function $A(x) = \int_a^x f(t)\,dt$.*

b. *Verify that $A'(x) = f(x)$.*



19. $f(t) = t + 5,\ a = -5$

20. $f(t) = 2t + 5,\ a = 0$

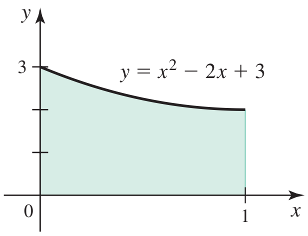
21. $f(t) = 3t + 1,\ a = 2$

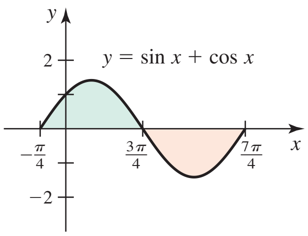
22. $f(t) = 4t + 2,\ a = 0$

23–24. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Explain why your result is consistent with the figure.*

23. $\displaystyle\int_0^1 (x^2 - 2x + 3)\,dx$

24. $\displaystyle\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\,dx$





25–28. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Sketch the graph of the integrand and shade the region whose net area you have found.*

25. $\displaystyle\int_{-2}^{3} (x^2 - x - 6)\,dx$

26. $\displaystyle\int_0^1 (x - \sqrt{x})\,dx$

27. $\displaystyle\int_0^5 (x^2 - 9)\,dx$

28. $\displaystyle\int_{1/2}^{2} \left(1 - \frac{1}{x^2}\right) dx$

29–50. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus.*

29. $\displaystyle\int_0^2 4x^3\,dx$

30. $\displaystyle\int_0^2 (3x^2 + 2x)\,dx$

31. $\displaystyle\int_0^1 (x + \sqrt{x})\,dx$

32. $\displaystyle\int_0^{\pi/4} 2\cos x\,dx$

33. $\displaystyle\int_1^9 \frac{2}{\sqrt{x}}\,dx$

34. $\displaystyle\int_4^9 \frac{2 + \sqrt{t}}{t}\,dt$

35. $\displaystyle\int_{-2}^{2} (x^2 - 4)\,dx$

36. $\displaystyle\int_0^{\ln 8} e^x\,dx$

37. $\displaystyle\int_{1/2}^{1} (x^{-3} - 8)\,dx$

38. $\displaystyle\int_0^4 x(x - 2)(x - 4)\,dx$

39. $\displaystyle\int_0^{\pi/4} \sec^2\theta\,d\theta$

40. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}$

41. $\displaystyle\int_{-2}^{-1} x^{-3}\,dx$

42. $\displaystyle\int_0^{\pi} (1 - \sin x)\,dx$

43. $\displaystyle\int_1^4 (1 - x)(x - 4)\,dx$

44. $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos x - 1)\,dx$

45. $\displaystyle\int_1^2 \frac{3}{t}\,dt$

46. $\displaystyle\int_4^9 \frac{x - \sqrt{x}}{x^3}\,dx$

47. $\displaystyle\int_0^{\pi/8} \cos 2x\,dx$

48. $\displaystyle\int_0^1 10e^{2x}\,dx$

49. $\displaystyle\int_1^{\sqrt{3}} \frac{dx}{1 + x^2}$

50. $\displaystyle\int_{\pi/16}^{\pi/8} 8\csc^2 4x\,dx$

51–54. Areas *Find (i) the net area and (ii) the area of the following regions. Graph the function and indicate the region in question.*

51. The region bounded by $y = x^{1/2}$ and the x-axis between $x = 1$ and $x = 4$

52. The region above the x-axis bounded by $y = 4 - x^2$

53. The region below the x-axis bounded by $y = x^4 - 16$

54. The region bounded by $y = 6\cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi$

55–60. Areas of regions *Find the area of the region bounded by the graph of f and the x-axis on the given interval.*

55. $f(x) = x^2 - 25$ on $[2, 4]$

56. $f(x) = x^3 - 1$ on $[-1, 2]$

57. $f(x) = \dfrac{1}{x}$ on $[-2, -1]$

58. $f(x) = x(x + 1)(x - 2)$ on $[-1, 2]$

59. $f(x) = \sin x$ on $[-\pi/4, 3\pi/4]$

60. $f(x) = \cos x$ on $[\pi/2, \pi]$

61–68. Derivatives of integrals *Simplify the following expressions.*

61. $\displaystyle\frac{d}{dx}\int_3^x (t^2 + t + 1)\,dt$

62. $\displaystyle\frac{d}{dx}\int_0^x e^t\,dt$

63. $\displaystyle\frac{d}{dx}\int_2^{x^3} \frac{dp}{p^2}$

64. $\displaystyle\frac{d}{dx}\int_{x^2}^{10} \frac{dz}{z^2 + 1}$

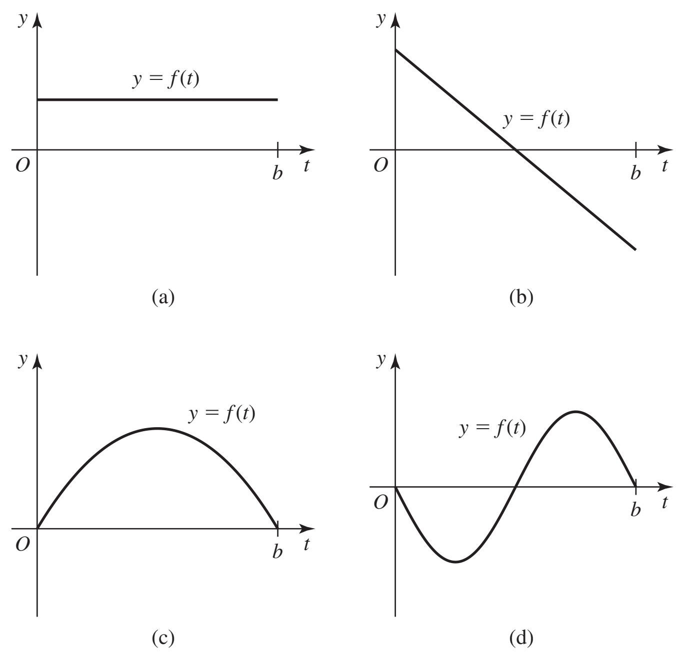
65. $\displaystyle\frac{d}{dx}\int_x^1 \sqrt{t^4 + 1}\,dt$

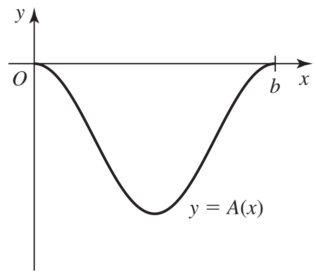
66. $\displaystyle\frac{d}{dx}\int_x^0 \frac{dp}{p^2 + 1}$

67. $\displaystyle\frac{d}{dx}\int_{-x}^{x} \sqrt{1 + t^2}\,dt$

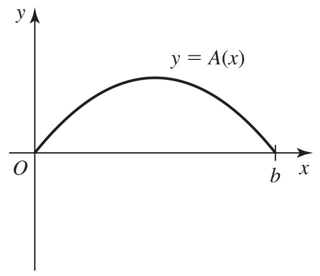
68. $\displaystyle\frac{d}{dx}\int_{e^x}^{e^{2x}} \ln t^2\,dt$

69. Matching functions with area functions Match the functions f, whose graphs are given in a–d, with the area functions $A(x) = \int_0^x f(t)\,dt$, whose graphs are given in A–D.





(A)



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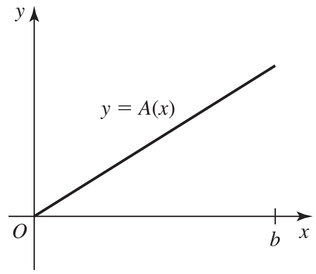

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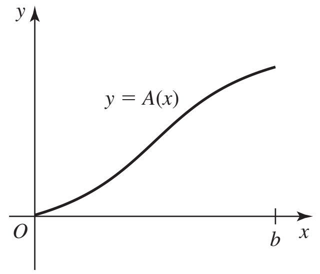


(D)

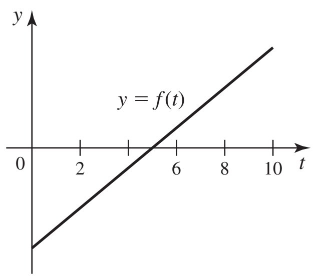
70–73. Working with area functions *Consider the function f and its graph.*

a. *Estimate the zeros of the area function* $A(x) = \int_0^x f(t)\,dt$*, for* $0 \le x \le 10$.

b. *Estimate the points (if any) at which A has a local maximum or minimum.*

c. *Sketch a graph of A, for* $0 \le x \le 10$*, without a scale on the y-axis.*

70.



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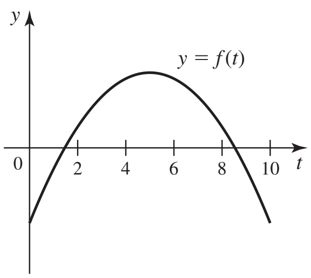

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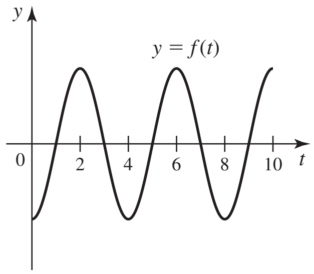

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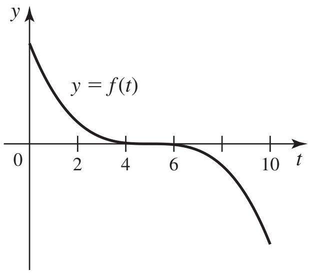
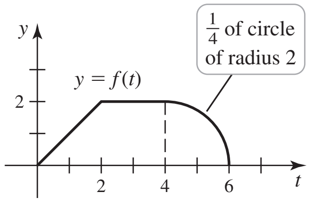
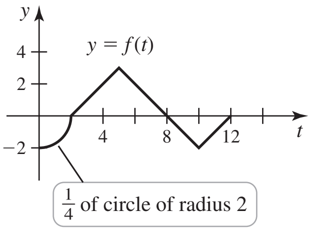


74. Area functions from graphs The graph of f is given in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and evaluate $A(1)$, $A(2)$, $A(4)$, and $A(6)$.



75. Area functions from graphs The graph of f is given in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and evaluate $A(2)$, $A(5)$, $A(8)$, and $A(12)$.



76–80. Working with area functions *Consider the function f and the points a, b, and c.*

a. *Find the area function* $A(x) = \int_a^x f(t)\,dt$ *using the Fundamental Theorem.*

b. *Graph f and A.*

c. *Evaluate* $A(b)$ *and* $A(c)$*. Interpret the results using the graphs of part (b).*

76. $f(x) = \sin x;\ a = 0, b = \pi/2, c = \pi$

77. $f(x) = e^x;\ a = 0, b = \ln 2, c = \ln 4$

78. $f(x) = -12x(x-1)(x-2);\ a = 0, b = 1, c = 2$

79. $f(x) = \cos \pi x;\ a = 0, b = \frac{1}{2}, c = 1$

80. $f(x) = \dfrac{1}{x};\ a = 1, b = 4, c = 6$

T **81–84. Functions defined by integrals** *Consider the function g, which is given in terms of a definite integral with a variable upper limit.*

a. *Graph the integrand.*

b. *Calculate* $g'(x)$.

c. *Graph g, showing all your work and reasoning.*

81. $g(x) = \displaystyle\int_0^x \sin^2 t\,dt$

82. $g(x) = \displaystyle\int_0^x (t^2 + 1)\,dt$

83. $g(x) = \displaystyle\int_0^x \sin(\pi t^2)\,dt$ (a Fresnel integral)

84. $g(x) = \displaystyle\int_0^x \cos(\pi\sqrt{t})\,dt$

Further Explorations

85. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Suppose that f is a positive decreasing function, for $x > 0$. Then the area function $A(x) = \int_0^x f(t)\,dt$ is an increasing function of x.

b. Suppose that f is a negative increasing function, for $x > 0$. Then the area function $A(x) = \int_0^x f(t)\,dt$ is a decreasing function of x.

c. The functions $p(x) = \sin 3x$ and $q(x) = 4 \sin 3x$ are antiderivatives of the same function.

d. If $A(x) = 3x^2 - x - 3$ is an area function for f, then $B(x) = 3x^2 - x$ is also an area function for f.

e. $\dfrac{d}{dx}\displaystyle\int_a^b f(t)\,dt = 0.$

86–94. Definite integrals *Evaluate the following definite integrals using the Fundamental Theorem of Calculus.*

86. $\frac{1}{2}\int_0^{\ln 2} e^x\,dx$ **87.** $\int_1^4 \frac{x-2}{\sqrt{x}}\,dx$ **88.** $\int_1^2 \left(\frac{2}{s} - \frac{4}{s^3}\right) ds$

89. $\int_0^{\pi/3} \sec x \tan x\,dx$ **90.** $\int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta$ **91.** $\int_1^8 \sqrt[3]{y}\,dy$

92. $\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}}$ **93.** $\int_1^2 \frac{z^2+4}{z}\,dz$ **94.** $\int_0^{\sqrt{3}} \frac{3\,dx}{9+x^2}$

T 95–98. Areas of regions *Find the area of the region R bounded by the graph of f and the x-axis on the given interval. Graph f and show the region R.*

95. $f(x) = 2 - |x|$ on $[-2, 4]$

96. $f(x) = (1 - x^2)^{-1/2}$ on $[-1/2, \sqrt{3}/2]$

97. $f(x) = x^4 - 4$ on $[1, 4]$ **98.** $f(x) = x^2(x-2)$ on $[-1, 3]$

99–102. Derivatives and integrals *Simplify the given expressions.*

99. $\int_3^8 f'(t)\,dt$, where f' is continuous on $[3, 8]$

100. $\frac{d}{dx}\int_0^{x^2} \frac{dt}{t^2+4}$ **101.** $\frac{d}{dx}\int_0^{\cos x} (t^4+6)\,dt$

102. $\frac{d}{dx}\int_x^1 e^{t^2}\,dt$ **103.** $\frac{d}{dt}\left(\int_1^t \frac{3}{x}\,dx - \int_{t^2}^1 \frac{3}{x}\,dx\right)$

104. $\frac{d}{dt}\left(\int_0^t \frac{dx}{1+x^2} + \int_0^{1/t} \frac{dx}{1+x^2}\right)$

Additional Exercises

105. Zero net area Consider the function $f(x) = x^2 - 4x$.

a. Graph f on the interval $x \geq 0$.
b. For what value of $b > 0$ is $\int_0^b f(x)\,dx = 0$?
c. In general, for the function $f(x) = x^2 - ax$, where $a > 0$, for what value of $b > 0$ (as a function of a) is $\int_0^b f(x)\,dx = 0$?

106. Cubic zero net area Consider the graph of the cubic $y = x(x-a)(x-b)$, where $0 < a < b$. Verify that the graph bounds a region above the x-axis, for $0 < x < a$, and bounds a region below the x-axis, for $a < x < b$. What is the relationship between a and b if the areas of these two regions are equal?

107. Maximum net area What value of $b > -1$ maximizes the integral

$$\int_{-1}^b x^2(3-x)\,dx?$$

108. Maximum net area Graph the function $f(x) = 8 + 2x - x^2$ and determine the values of a and b that maximize the value of the integral

$$\int_a^b (8 + 2x - x^2)\,dx.$$

109. An integral equation Use the Fundamental Theorem of Calculus, Part 1, to find the function f that satisfies the equation

$$\int_0^x f(t)\,dt = 2\cos x + 3x - 2.$$

Verify the result by substitution into the equation.

110. Max/min of area functions Suppose f is continuous on $[0, \infty)$ and $A(x)$ is the net area of the region bounded by the graph of f and the t-axis on $[0, x]$. Show that the local maxima and minima of A occur at the zeros of f. Verify this fact with the function $f(x) = x^2 - 10x$.

T 111. Asymptote of sine integral Use a calculator to approximate

$$\lim_{x\to\infty} S(x) = \lim_{x\to\infty} \int_0^x \frac{\sin t}{t}\,dt,$$

where S is the sine integral function (see Example 7). Explain your reasoning.

112. Sine integral Show that the sine integral $S(x) = \int_0^x \frac{\sin t}{t}\,dt$ satisfies the (differential) equation $xS'(x) + 2S''(x) + xS'''(x) = 0$.

113. Fresnel integral Show that the Fresnel integral $S(x) = \int_0^x \sin t^2\,dt$ satisfies the (differential) equation

$$(S'(x))^2 + \left(\frac{S''(x)}{2x}\right)^2 = 1.$$

114. Variable integration limits Evaluate $\frac{d}{dx}\int_{-x}^x (t^2 + t)\,dt$.
(*Hint:* Separate the integral into two pieces.)

115. Discrete version of the Fundamental Theorem In this exercise, we work with a discrete problem and show why the relationship $\int_a^b f'(x)\,dx = f(b) - f(a)$ makes sense. Suppose we have a set of equally spaced grid points

$$\{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\},$$

where the distance between any two grid points is Δx. Suppose also that at each grid point x_k, a function value $f(x_k)$ is defined, for $k = 0, \ldots, n$.

a. We now replace the integral with a sum and replace the derivative with a difference quotient. Explain why $\int_a^b f'(x)\,dx$ is analogous to $\sum_{k=1}^{n} \underbrace{\frac{f(x_k) - f(x_{k-1})}{\Delta x}}_{\approx f'(x_k)} \Delta x$.
b. Simplify the sum in part (a) and show that it is equal to $f(b) - f(a)$.
c. Explain the correspondence between the integral relationship and the summation relationship.

116. Continuity at the endpoints Assume that f is continuous on $[a, b]$ and let A be the area function for f with left endpoint a. Let m^* and M^* be the absolute minimum and maximum values of f on $[a, b]$, respectively.

a. Prove that $m^*(x-a) \leq A(x) \leq M^*(x-a)$ for all x in $[a, b]$. Use this result and the Squeeze Theorem to show that A is continuous from the right at $x = a$.
b. Prove that $m^*(b-x) \leq A(b) - A(x) \leq M^*(b-x)$ for all x in $[a, b]$. Use this result to show that A is continuous from the left at $x = b$.

QUICK CHECK ANSWERS

1. $0, -35$ **2.** $A(6) = 44$; $A(10) = 120$ **3.** $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$
4. If f is differentiated, we get f'. Therefore, f is an antiderivative of f'. ◄

5.4 Working with Integrals

With the Fundamental Theorem of Calculus in hand, we may begin an investigation of integration and its applications. In this section, we discuss the role of symmetry in integrals, we use the slice-and-sum strategy to define the average value of a function, and we explore a theoretical result called the Mean Value Theorem for Integrals.

Integrating Even and Odd Functions

Symmetry appears throughout mathematics in many different forms, and its use often leads to insights and efficiencies. Here we use the symmetry of a function to simplify integral calculations.

Section 1.1 introduced the symmetry of even and odd functions. An **even function** satisfies the property $f(-x) = f(x)$, which means that its graph is symmetric about the y-axis (Figure 5.49a). Examples of even functions are $f(x) = \cos x$ and $f(x) = x^n$, where n is an even integer. An **odd function** satisfies the property $f(-x) = -f(x)$, which means that its graph is symmetric about the origin (Figure 5.49b). Examples of odd functions are $f(x) = \sin x$ and $f(x) = x^n$, where n is an odd integer.

Special things happen when we integrate even and odd functions on intervals centered at the origin. First suppose f is an even function and consider $\int_{-a}^{a} f(x)\,dx$. From Figure 5.49a, we see that the integral of f on $[-a, 0]$ equals the integral of f on $[0, a]$. Therefore, the integral on $[-a, a]$ is twice the integral on $[0, a]$, or

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

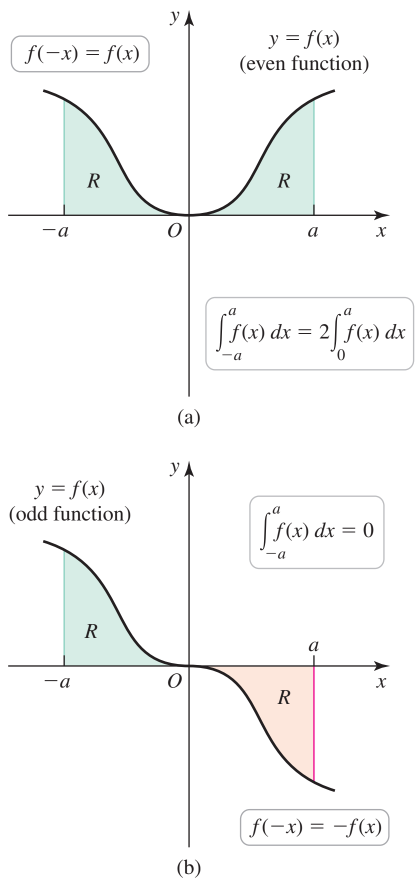


Figure 5.49

On the other hand, suppose f is an odd function and consider $\int_{-a}^{a} f(x)\,dx$. As shown in Figure 5.49b, the integral on the interval $[-a, 0]$ is the negative of the integral on $[0, a]$. Therefore, the integral on $[-a, a]$ is zero, or

$$\int_{-a}^{a} f(x)\,dx = 0.$$

We summarize these results in the following theorem.

THEOREM 5.4 Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

- If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
- If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

QUICK CHECK 1 If f and g are both even functions, is the product fg even or odd? Use the facts that $f(-x) = f(x)$ and $g(-x) = g(x)$. ◄

The following example shows how symmetry can simplify integration.

EXAMPLE 1 Integrating symmetric functions Evaluate the following integrals using symmetry arguments.

a. $\displaystyle\int_{-2}^{2} (x^4 - 3x^3)\,dx$ **b.** $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos x - 4\sin^3 x)\,dx$

SOLUTION

a. Note that $x^4 - 3x^3$ is neither odd nor even so Theorem 5.4 cannot be applied directly. However, we can split the integral and then use symmetry:

$$\int_{-2}^{2} (x^4 - 3x^3)\,dx = \int_{-2}^{2} x^4\,dx - 3\underbrace{\int_{-2}^{2} x^3\,dx}_{0} \quad \text{Properties 3 and 4 of Table 5.4}$$

$$= 2\int_{0}^{2} x^4\,dx - 0 \quad x^4 \text{ is even; } x^3 \text{ is odd.}$$

$$= 2\left(\frac{x^5}{5}\right)\Bigg|_0^2 \quad \text{Fundamental Theorem}$$

$$= 2\left(\frac{32}{5}\right) = \frac{64}{5}. \quad \text{Simplify.}$$

Notice how the odd-powered term of the integrand is eliminated by symmetry. Integration of the even-powered term is simplified because the lower limit is zero.

b. The $\cos x$ term is an even function, so it can be integrated on the interval $[0, \pi/2]$. What about $\sin^3 x$? It is an odd function raised to an odd power, which results in an odd function; its integral on $[-\pi/2, \pi/2]$ is zero. Therefore,

$$\int_{-\pi/2}^{\pi/2} (\cos x - 4\sin^3 x)\,dx = 2\int_{0}^{\pi/2} \cos x\,dx - 0 \quad \text{Symmetry}$$

$$= 2\sin x\Big|_0^{\pi/2} \quad \text{Fundamental Theorem}$$

$$= 2(1 - 0) = 2. \quad \text{Simplify.}$$

Related Exercises 7–20 ◀

➤ There are a couple of ways to see that $\sin^3 x$ is an odd function. Its graph is symmetric about the origin, indicating that $\sin^3(-x) = -\sin^3 x$. Or by analogy, take an odd power of x and raise it to an odd power. For example, $(x^5)^3 = x^{15}$, which is odd. See Exercises 53–56 for direct proofs of symmetry in composite functions.

Average Value of a Function

If five people weigh 155, 143, 180, 105, and 123 lb, their average (mean) weight is

$$\frac{155 + 143 + 180 + 105 + 123}{5} = 141.2 \text{ lb.}$$

This idea generalizes quite naturally to functions. Consider a function f that is continuous on $[a, b]$. Using a regular partition $x_0 = a, x_1, x_2, \ldots, x_n = b$ with $\Delta x = \frac{b - a}{n}$, we select a point x_k^* in each subinterval and compute $f(x_k^*)$, for $k = 1, \ldots, n$. The values of $f(x_k^*)$ may be viewed as a sampling of f on $[a, b]$. The average of these function values is

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n}.$$

Noting that $n = \frac{b - a}{\Delta x}$, we write the average of the n sample values as the Riemann sum

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{(b - a)/\Delta x} = \frac{1}{b - a}\sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Now suppose we increase n, taking more and more samples of f, while Δx decreases to zero. The limit of this sum is a definite integral that gives the average value $\bar{f}$ on $[a, b]$:

$$\bar{f} = \frac{1}{b - a}\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \frac{1}{b - a}\int_a^b f(x)\,dx.$$