GLOBAL EDITION

Calculus

SECOND EDITION

William Briggs • Lyle Cochran • Bernard Gillett

ALWAYS LEARNING

PEARSON

ALGEBRA

Exponents and Radicals

$x^a x^b = x^{a+b}$ $\quad \dfrac{x^a}{x^b} = x^{a-b}$ $\quad x^{-a} = \dfrac{1}{x^a}$ $\quad (x^a)^b = x^{ab}$ $\quad \left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$

$x^{1/n} = \sqrt[n]{x}$ $\quad x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$ $\quad \sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$ $\quad \sqrt[n]{x/y} = \sqrt[n]{x}/\sqrt[n]{y}$

Factoring Formulas

$a^2 - b^2 = (a - b)(a + b)$ $\quad a^2 + b^2$ does not factor over real numbers.

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ $\quad a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$

Binomials

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

Binomial Theorem

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n,$$

where $\dbinom{n}{k} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1} = \dfrac{n!}{k!(n-k)!}$

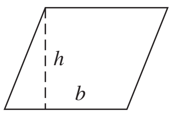
Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

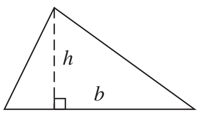
GEOMETRY

Parallelogram



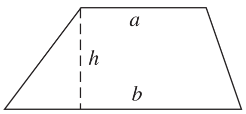
$A = bh$

Triangle



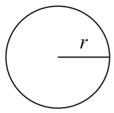
$A = \frac{1}{2}bh$

Trapezoid



$A = \frac{1}{2}(a + b)h$

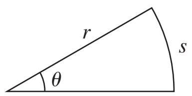
Circle



$A = \pi r^2$

$C = 2\pi r$

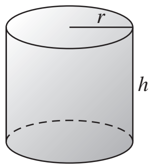
Sector



$A = \frac{1}{2}r^2\theta$

$s = r\theta$ (θ in radians)

Cylinder

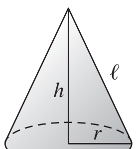


$V = \pi r^2 h$

$S = 2\pi rh$

(lateral surface area)

Cone

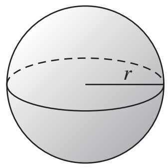


$V = \frac{1}{3}\pi r^2 h$

$S = \pi r\ell$

(lateral surface area)

Sphere

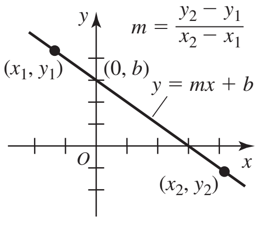


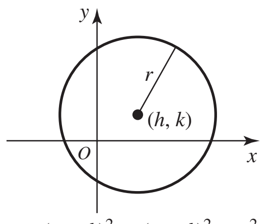
$V = \frac{4}{3}\pi r^3$

$S = 4\pi r^2$

Equations of Lines and Circles

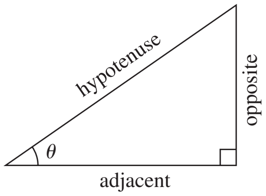
$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	slope of line through (x_1, y_1) and (x_2, y_2)
$y - y_1 = m(x - x_1)$	point–slope form of line through (x_1, y_1) with slope m
$y = mx + b$	slope–intercept form of line with slope m and y-intercept $(0, b)$
$(x - h)^2 + (y - k)^2 = r^2$	circle of radius r with center (h, k)



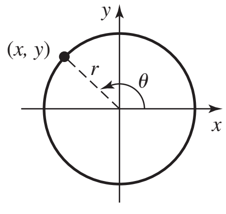


$(x - h)^2 + (y - k)^2 = r^2$

TRIGONOMETRY



$\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$	$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$	$\tan\theta = \dfrac{\text{opp}}{\text{adj}}$
$\sec\theta = \dfrac{\text{hyp}}{\text{adj}}$	$\csc\theta = \dfrac{\text{hyp}}{\text{opp}}$	$\cot\theta = \dfrac{\text{adj}}{\text{opp}}$



$\cos\theta = \dfrac{x}{r}$	$\sec\theta = \dfrac{r}{x}$
$\sin\theta = \dfrac{y}{r}$	$\csc\theta = \dfrac{r}{y}$
$\tan\theta = \dfrac{y}{x}$	$\cot\theta = \dfrac{x}{y}$

(*Continued*)

➤ Ignoring the factor of π, the integrand in the washer method integral is $f(x)^2 - g(x)^2$, which is not equal to $(f(x) - g(x))^2$.

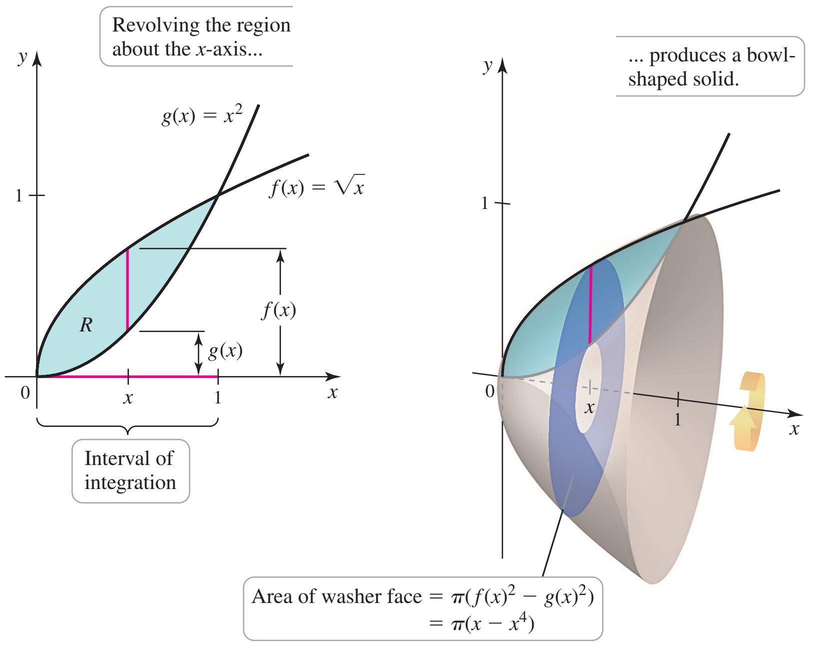


Figure 6.33

Related Exercises 25–32 ◄

QUICK CHECK 5 Suppose the region in Example 4 is revolved about the line $y = -1$ instead of the x-axis. (a) What is the inner radius of a typical washer? (b) What is the outer radius of a typical washer? ◄

Revolving about the y-Axis

Everything you learned about revolving regions about the x-axis applies to revolving regions about the y-axis. Consider a region R bounded by the curve $x = p(y)$ on the right, the curve $x = q(y)$ on the left, and the horizontal lines $y = c$ and $y = d$ (Figure 6.34a).

To find the volume of the solid generated when R is revolved about the y-axis, we use the general slicing method—now with respect to the y-axis (Figure 6.34b). The area of a typical cross section is $A(y) = \pi(p(y)^2 - q(y)^2)$, where $c \le y \le d$. As before, integrating these cross-sectional areas of the solid gives the volume.

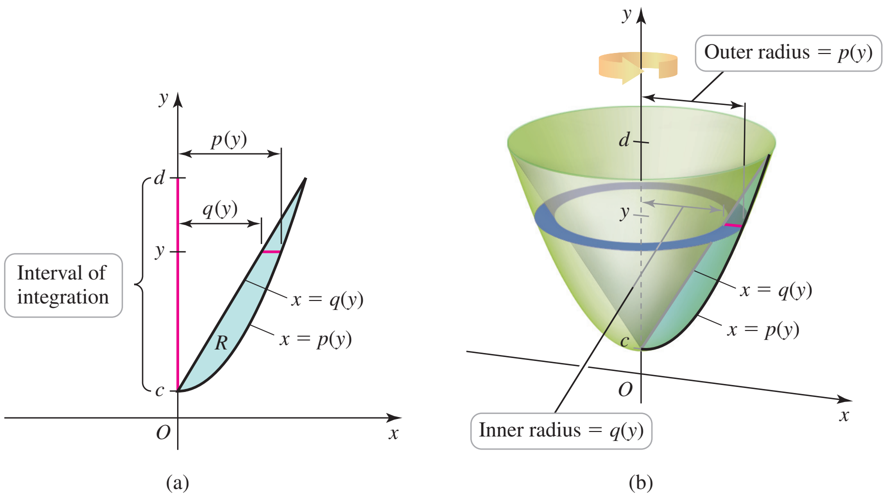


Figure 6.34

➤ The disk/washer method about the y-axis is the disk/washer method about the x-axis with x replaced with y.

Disk and Washer Methods about the y-Axis

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y)$, $x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi(p(y)^2 - q(y)^2)\, dy.$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi p(y)^2\, dy.$$

EXAMPLE 5 **Which solid has greater volume?** Let R be the region in the first quadrant bounded by the graphs of $x = y^3$ and $x = 4y$. Which is greater, the volume of the solid generated when R is revolved about the x-axis or the y-axis?

SOLUTION Solving $y^3 = 4y$, or equivalently, $y(y^2 - 4) = 0$, we find that the bounding curves of R intersect at the points $(0, 0)$ and $(8, 2)$. When the region R (Figure 6.35a) is revolved about the y-axis, it generates a funnel with a curved inner surface (Figure 6.35b). Washer-shaped cross sections perpendicular to the y-axis extend from $y = 0$ to $y = 2$. The outer radius of the cross section at the point y is determined by the line $x = p(y) = 4y$. The inner radius of the cross section at the point y is determined by the curve $x = q(y) = y^3$. Applying the washer method, the volume of this solid is

$$\begin{aligned} V &= \int_0^2 \pi(p(y)^2 - q(y)^2)\, dy && \text{Washer method} \\ &= \int_0^2 \pi(16y^2 - y^6)\, dy && \text{Substitute for } p \text{ and } q. \\ &= \pi\left(\frac{16}{3}y^3 - \frac{y^7}{7}\right)\Bigg|_0^2 && \text{Fundamental Theorem} \\ &= \frac{512\pi}{21}. && \text{Evaluate.} \end{aligned}$$

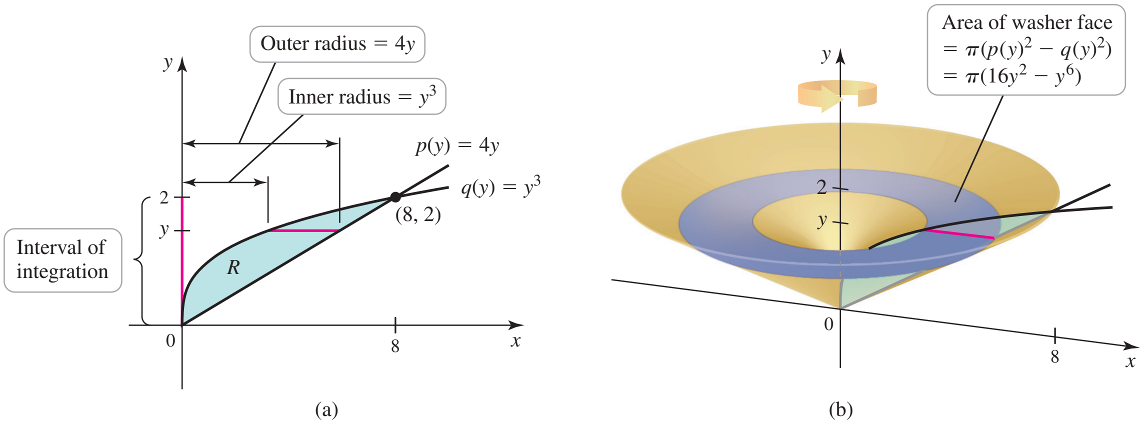


Figure 6.35

When the region R is revolved about the x-axis, it generates a different funnel (Figure 6.36). Vertical slices through the solid between $x = 0$ and $x = 8$ produce washers. The outer radius of the washer at the point x is determined by the curve $x = y^3$, or

$y = f(x) = x^{1/3}$. The inner radius is determined by $x = 4y$, or $y = g(x) = x/4$. The volume of the resulting solid is

$$\begin{aligned} V &= \int_0^8 \pi(f(x)^2 - g(x)^2)\,dx && \text{Washer method} \\ &= \int_0^8 \pi\left(x^{2/3} - \frac{x^2}{16}\right) dx && \text{Substitute for } f \text{ and } g. \\ &= \pi\left(\frac{3}{5}x^{5/3} - \frac{x^3}{48}\right)\Bigg|_0^8 && \text{Fundamental Theorem} \\ &= \frac{128\pi}{15}. && \text{Evaluate.} \end{aligned}$$

We see that revolving the region about the y-axis produces a solid of greater volume.

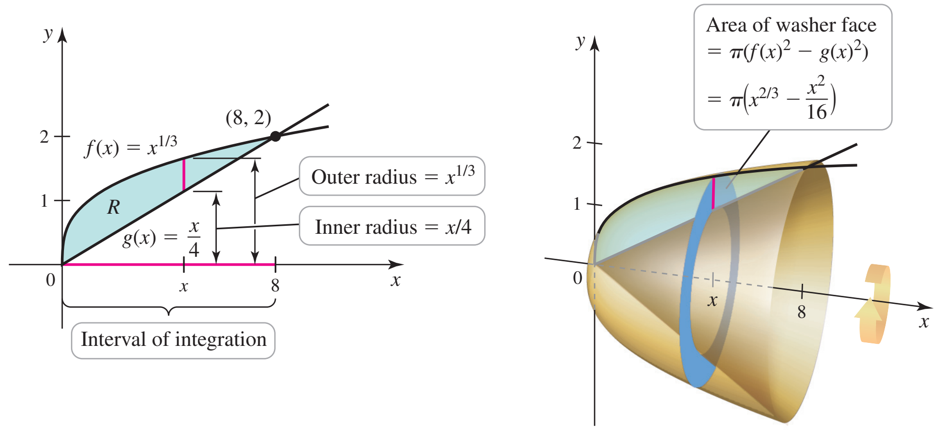


Figure 6.36

Related Exercises 33–42 ◄

QUICK CHECK 6 The region in the first quadrant bounded by $y = x$ and $y = x^3$ is revolved about the y-axis. Give the integral for the volume of the solid that is generated. ◄

The disk and washer methods may be generalized to handle situations in which a region R is revolved about a line parallel to one of the coordinate axes. The next example discusses three such cases.

EXAMPLE 6 **Revolving about other lines** Let $f(x) = \sqrt{x} + 1$ and $g(x) = x^2 + 1$.

a. Find the volume of the solid generated when the region R_1 bounded by the graph of f and the line $y = 2$ on the interval $[0, 1]$ is revolved about the line $y = 2$.

b. Find the volume of the solid generated when the region R_2 bounded by the graphs of f and g on the interval $[0, 1]$ is revolved about the line $y = -1$.

c. Find the volume of the solid generated when the region R_2 bounded by the graphs of f and g on the interval $[0, 1]$ is revolved about the line $x = 2$.

SOLUTION

a. Figure 6.37a shows the region R_1 and the axis of revolution. Applying the disk method, we see that a disk located at a point x has a radius of $2 - f(x) = 2 - (\sqrt{x} + 1) = 1 - \sqrt{x}$. Therefore, the volume of the solid generated when R_1 is revolved about $y = 2$ is

$$\int_0^1 \pi(1 - \sqrt{x})^2\,dx = \pi\int_0^1 (1 - 2\sqrt{x} + x)\,dx = \frac{\pi}{6}.$$

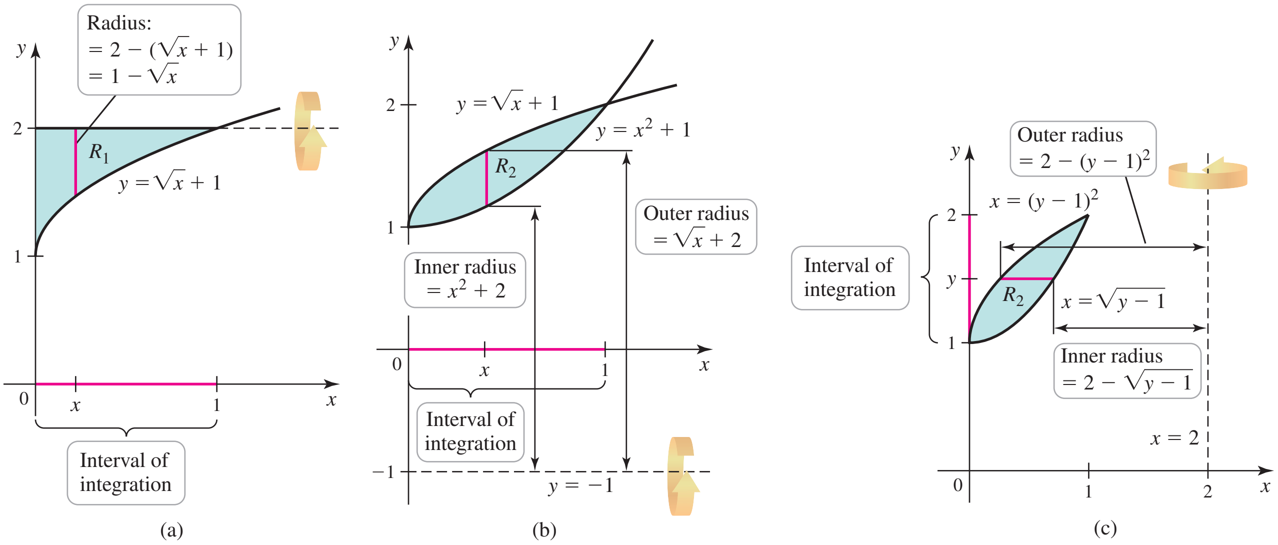


Figure 6.37

b. When the graph of f is revolved about $y = -1$, it sweeps out a solid of revolution whose radius at a point x is $f(x) + 1 = \sqrt{x} + 2$. Similarly, when the graph of g is revolved about $y = -1$, it sweeps out a solid of revolution whose radius at a point x is $g(x) + 1 = x^2 + 2$ (Figure 6.37b). Using the washer method, the volume of the solid generated when R_2 is revolved about $y = -1$ is

$$\int_0^1 \pi\left((\sqrt{x} + 2)^2 - (x^2 + 2)^2\right) dx$$

$$= \pi \int_0^1 (-x^4 - 4x^2 + x + 4\sqrt{x})\, dx$$

$$= \frac{49\pi}{30}.$$

c. When the region R_2 is revolved about the line $x = 2$, we use the washer method and integrate in the y-direction. First note that the graph of f is described by $y = \sqrt{x} + 1$, or equivalently, $x = (y - 1)^2$, for $y \geq 1$. Also, the graph of g is described by $y = x^2 + 1$, or equivalently, $x = \sqrt{y - 1}$ for $y \geq 1$ (Figure 6.37c). When the graph of f is revolved about the line $x = 2$, the radius of a typical disk at a point y is $2 - (y - 1)^2$. Similarly, when the graph of g is revolved about $x = 2$, the radius of a typical disk at a point y is $2 - \sqrt{y - 1}$. Finally, observe that the extent of the region R_2 in the y-direction is the interval $1 \leq y \leq 2$.

Applying the washer method, simplifying the integrand, and integrating powers of y, the volume of the solid of revolution is

$$\int_1^2 \pi\left((2 - (y - 1)^2)^2 - (2 - \sqrt{y - 1})^2\right) dy = \frac{31\pi}{30}.$$

Related Exercises 43–50 ◄

SECTION 6.3 EXERCISES

Review Questions

1. Suppose a cut is made through a solid object perpendicular to the x-axis at a particular point x. Explain the meaning of $A(x)$.

2. A solid has a circular base and cross sections perpendicular to the base are squares. What method should be used to find the volume of the solid?

3. The region bounded by the curves $y = 2x$ and $y = x^2$ is revolved about the x-axis. Give an integral for the volume of the solid that is generated.

4. The region bounded by the curves $y = 2x$ and $y = x^2$ is revolved about the y-axis. Give an integral for the volume of the solid that is generated.

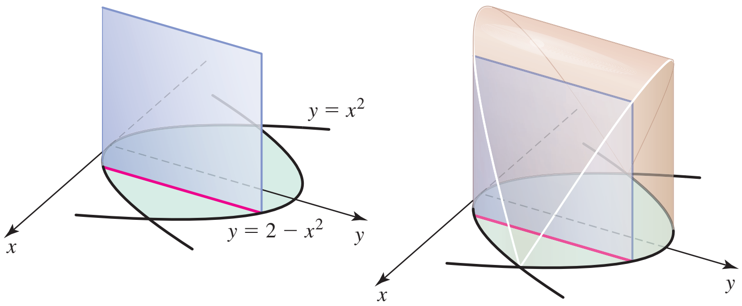
5. Why is the disk method a special case of the general slicing method?

6. The region R bounded by the graph of $y = f(x) \geq 0$ and the x-axis on $[a, b]$ is revolved about the line $y = -2$ to form a solid of revolution whose cross sections are washers. What are the inner and outer radii of the washer at a point x in $[a, b]$?

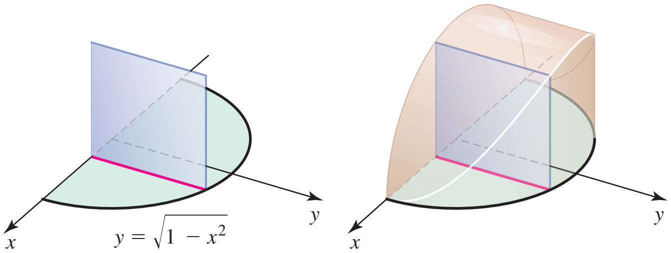
Basic Skills

7–16. General slicing method *Use the general slicing method to find the volume of the following solids.*

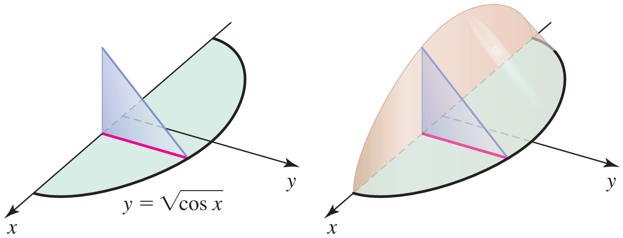
7. The solid whose base is the region bounded by the curves $y = x^2$ and $y = 2 - x^2$, and whose cross sections through the solid perpendicular to the x-axis are squares



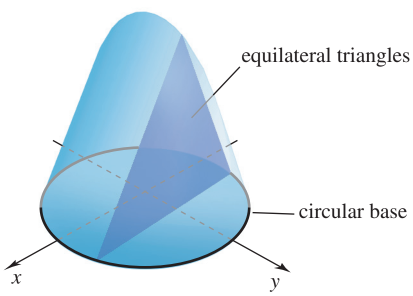
8. The solid whose base is the region bounded by the semicircle $y = \sqrt{1 - x^2}$ and the x-axis, and whose cross sections through the solid perpendicular to the x-axis are squares



9. The solid whose base is the region bounded by the curve $y = \sqrt{\cos x}$ and the x-axis on $[-\pi/2, \pi/2]$, and whose cross sections through the solid perpendicular to the x-axis are isosceles right triangles with a horizontal leg in the xy-plane and a vertical leg above the x-axis

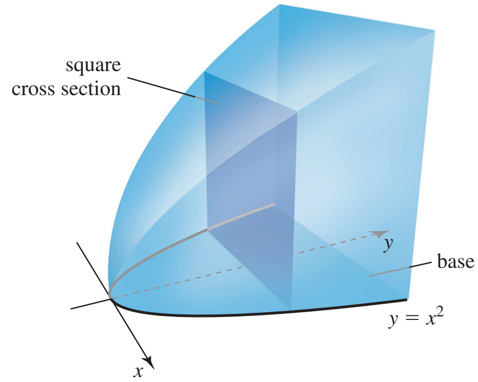


10. The solid with a circular base of radius 5 whose cross sections perpendicular to the base and parallel to the x-axis are equilateral triangles



11. The solid with a semicircular base of radius 5 whose cross sections perpendicular to the base and parallel to the diameter are squares

12. The solid whose base is the region bounded by $y = x^2$ and the line $y = 1$, and whose cross sections perpendicular to the base and parallel to the x-axis are squares

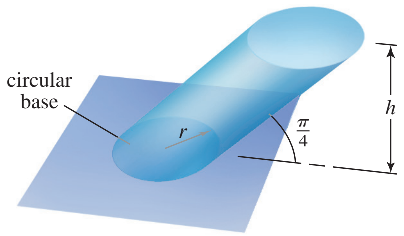


13. The solid whose base is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 2)$, and whose cross sections perpendicular to the base and parallel to the y-axis are semicircles

14. The pyramid with a square base 4 m on a side and a height of 2 m (Use calculus.)

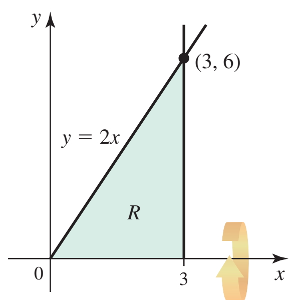
15. The tetrahedron (pyramid with four triangular faces), all of whose edges have length 4

16. A circular cylinder of radius r and height h whose axis is at an angle of $\pi/4$ to the base

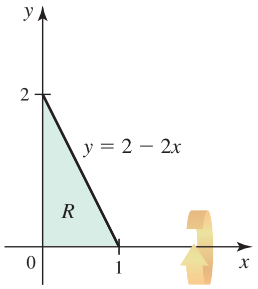


17–24. Disk method *Let R be the region bounded by the following curves. Use the disk method to find the volume of the solid generated when R is revolved about the x-axis.*

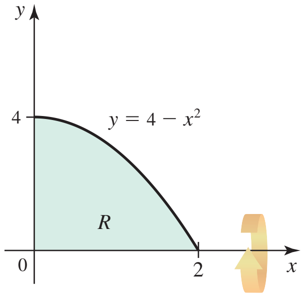
17. $y = 2x$, $y = 0$, $x = 3$ (Verify that your answer agrees with the volume formula for a cone.)



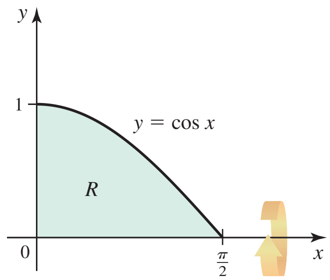
18. $y = 2 - 2x, y = 0, x = 0$ (Verify that your answer agrees with the volume formula for a cone.)



19. $y = 4 - x^2, y = 0, x = 0$



20. $y = \cos x$ on $[0, \pi/2], y = 0, x = 0$ (Recall that $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.)



21. $y = \sin x$ on $[0, \pi], y = 0$ (Recall that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.)

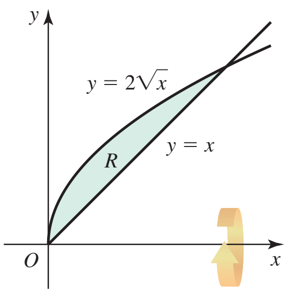
22. $y = \sqrt{25 - x^2}, y = 0$ (Verify that your answer agrees with the volume formula for a sphere.)

23. $y = \dfrac{1}{x^2}, y = 0, x = 1$, and $x = 4$

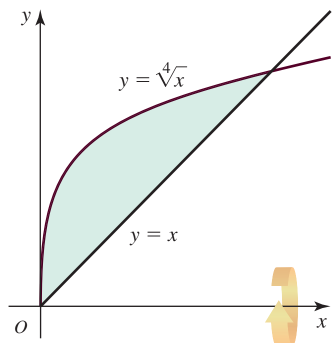
24. $y = \sec x, y = 0, x = 0$, and $x = \frac{\pi}{4}$

25–32. Washer method *Let R be the region bounded by the following curves. Use the washer method to find the volume of the solid generated when R is revolved about the x-axis.*

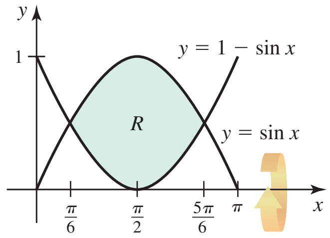
25. $y = x, y = 2\sqrt{x}$



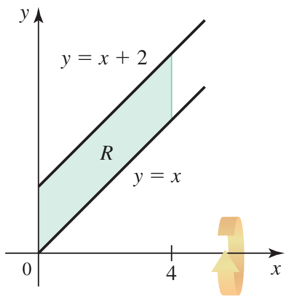
26. $y = x, y = \sqrt[4]{x}$



27. $y = \sin x, y = 1 - \sin x, x = \pi/6, x = 5\pi/6$



28. $y = x, y = x + 2, x = 0, x = 4$



29. $y = x + 3, y = x^2 + 1$

30. $y = \sqrt{\sin x}, y = 1, x = 0$

31. $y = \sin x, y = \sqrt{\sin x}$, for $0 \le x \le \pi/2$

32. $y = |x|, y = 2 - x^2$

33–38. Disks/washers about the *y*-axis *Let R be the region bounded by the following curves. Use the disk or washer method to find the volume of the solid generated when R is revolved about the y-axis.*

33. $y = x, y = 2x, y = 6$