GLOBAL EDITION

Nonlinear Control

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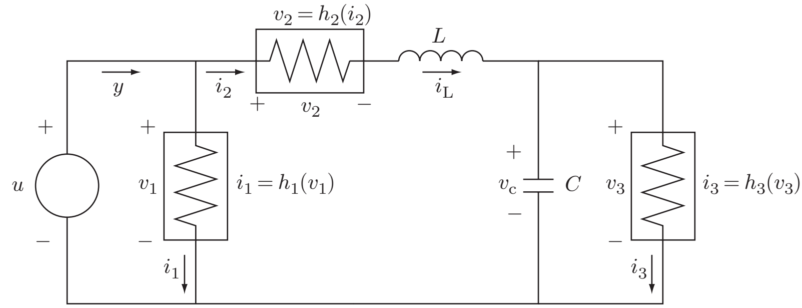


Figure 5.6: RLC circuit of Example 5.1.

Example 5.1 The *RLC* circuit of Figure 5.6 features a voltage source connected to an *RLC* network with linear inductor and capacitor and nonlinear resistors. The nonlinear resistors 1 and 3 are represented by their v–i characteristics $i_1 = h_1(v_1)$ and $i_3 = h_3(v_3)$, while resistor 2 is represented by its i–v characteristic $v_2 = h_2(i_2)$. We take the voltage u as the input and the current y as the output. The product uy is the power flow into the network. Taking the current x_1 through the inductor and the voltage x_2 across the capacitor as the state variables, the state model is

$$L\dot{x}_1 = u - h_2(x_1) - x_2, \quad C\dot{x}_2 = x_1 - h_3(x_2), \quad y = x_1 + h_1(u)$$

The new feature of an *RLC* network over a resistive network is the presence of the energy-storing elements L and C. The system is passive if the energy absorbed by the network over any period of time $[0, t]$ is greater than or equal to the change in the energy stored in the network over the same period; that is,

$$\int_0^t u(s)y(s)\ ds \geq V(x(t)) - V(x(0)) \tag{5.7}$$

where $V(x) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$ is the energy stored in the network. If (5.7) holds with strict inequality, then the difference between the absorbed energy and the change in the stored energy must be the energy dissipated in the resistors. Since (5.7) must hold for every $t \geq 0$, the instantaneous power inequality

$$u(t)y(t) \geq \dot{V}(x(t), u(t)) \tag{5.8}$$

must hold for all t; that is, the power flow into the network must be greater than or equal to the rate of change of the energy stored in the network. We can investigate inequality (5.8) by calculating the derivative of V along the trajectories of the system. We have

$$\begin{aligned}
\dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 = x_1[u - h_2(x_1) - x_2] + x_2[x_1 - h_3(x_2)] \\
&= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\
&= [x_1 + h_1(u)]u - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\
&= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2)
\end{aligned}$$

Thus,

$$uy = \dot{V} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2)$$

If h_1, h_2, and h_3 are passive, $uy \geq \dot{V}$ and the system is passive. Other possibilities are illustrated by four different special cases of the network.

Case 1: If $h_1 = h_2 = h_3 = 0$, then $uy = \dot{V}$ and there is no energy dissipation in the network; that is, the system is lossless.

Case 2: If h_2 and h_3 are passive, then $uy \geq \dot{V} + uh_1(u)$. If $uh_1(u) > 0$ for all $u \neq 0$, the energy absorbed over $[0, t]$ will be equal to the change in the stored energy only when $u(t) \equiv 0$. This is a case of input strict passivity.

Case 3: If $h_1 = 0$ and h_3 is passive, then $uy \geq \dot{V} + yh_2(y)$. When $yh_2(y) > 0$ for all $y \neq 0$, we have output strict passivity because the energy absorbed over $[0, t]$ will be equal to the change in the stored energy only when $y(t) \equiv 0$.

Case 4: If $h_1 \in [0, \infty]$, $h_2 \in (0, \infty)$, and $h_3 \in (0, \infty)$, then

$$uy \geq \dot{V} + x_1h_2(x_1) + x_2h_3(x_2)$$

where $x_1h_2(x_1) + x_2h_3(x_2)$ is a positive definite function of x. This is a case of state strict passivity because the energy absorbed over $[0, t]$ will be equal to the change in the stored energy only when $x(t) \equiv 0$. A system having this property is called *state strictly passive* or, simply, *strictly passive*.

△

Definition 5.3 *The system* (5.6) *is passive if there exists a continuously differentiable positive semidefinite function* $V(x)$ (*called the storage function*) *such that*

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \quad \forall\ (x, u) \tag{5.9}$$

Moreover, it is

- *lossless if* $u^T y = \dot{V}$.
- *input strictly passive if* $u^T y \geq \dot{V} + u^T \varphi(u)$ *and* $u^T \varphi(u) > 0,\ \forall\ u \neq 0$.
- *output strictly passive if* $u^T y \geq \dot{V} + y^T \rho(y)$ *and* $y^T \rho(y) > 0,\ \forall\ y \neq 0$.
- *strictly passive if* $u^T y \geq \dot{V} + \psi(x)$ *for some positive definite function* ψ.

In all cases, the inequality should hold for all (x, u).

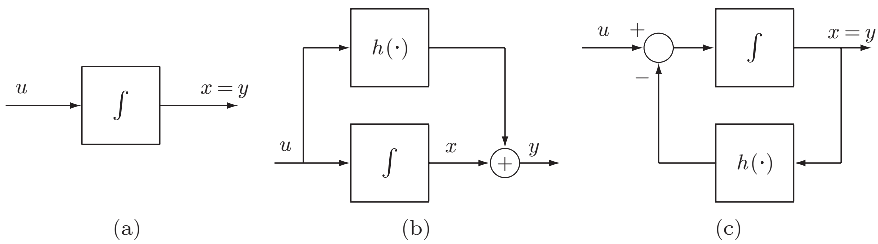


Figure 5.7: Example 5.2

Example 5.2 The integrator of Figure 5.7(a), represented by

$$\dot{x} = u, \qquad y = x$$

is a lossless system since, with $V(x) = \frac{1}{2}x^2$ as the storage function, $uy = \dot{V}$. When a memoryless function is connected in parallel with the integrator, as shown in Figure 5.7(b), the system is represented by

$$\dot{x} = u, \qquad y = x + h(u)$$

With $V(x) = \frac{1}{2}x^2$ as the storage function, $uy = \dot{V} + uh(u)$. If $h \in [0, \infty]$, the system is passive. If $uh(u) > 0$ for all $u \neq 0$, the system is input strictly passive. When the loop is closed around the integrator with a memoryless function, as in Figure 5.7(c), the system is represented by

$$\dot{x} = -h(x) + u, \qquad y = x$$

With $V(x) = \frac{1}{2}x^2$ as the storage function, $uy = \dot{V} + yh(y)$. If $h \in [0, \infty]$, the system is passive. If $yh(y) > 0$ for all $y \neq 0$, the system is output strictly passive. △

Example 5.3 The cascade connection of an integrator and a passive memoryless function, shown in Figure 5.8(a), is represented by

$$\dot{x} = u, \qquad y = h(x)$$

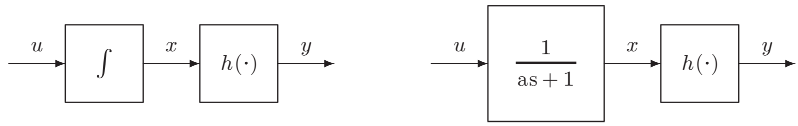
Passivity of h guarantees that $\int_0^x h(\sigma)\, d\sigma \geq 0$ for all x. With $V(x) = \int_0^x h(\sigma)\, d\sigma$ as the storage function, $\dot{V} = h(x)\dot{x} = yu$. Hence, the system is lossless. Suppose now the integrator is replaced by the transfer function $1/(as + 1)$ with $a > 0$, as shown in Figure 5.8(b). The system can be represented by the state model

$$a\dot{x} = -x + u, \qquad y = h(x)$$

With $V(x) = a \int_0^x h(\sigma)\, d\sigma$ as the storage function,

$$\dot{V} = h(x)(-x + u) = yu - xh(x) \leq yu$$

Hence, the system is passive. If $xh(x) > 0 \ \forall\ x \neq 0$, it is strictly passive. △



(a) (b)

Figure 5.8: Example 5.3

Example 5.4 Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2 + u, \qquad y = bx_2 + u$$

where $h \in [\alpha_1, \infty]$, and a, b, α_1 are positive constants. Let

$$V(x) = \alpha \int_0^{x_1} h(\sigma)\, d\sigma + \tfrac{1}{2}\alpha x^T P x = \alpha \int_0^{x_1} h(\sigma)\, d\sigma + \tfrac{1}{2}\alpha(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2)$$

where α, p_{11}, and $p_{11}p_{22} - p_{12}^2$ are positive. Use V as a storage function candidate.

$$uy - \dot{V} = u(bx_2 + u) - \alpha[h(x_1) + p_{11}x_1 + p_{12}x_2]x_2 - \alpha(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + u]$$

Take $p_{22} = 1$, $p_{11} = ap_{12}$, and $\alpha = b$ to cancel the cross product terms $x_2h(x_1)$, x_1x_2, and x_2u, respectively. Then,

$$\begin{aligned} uy - \dot{V} &= u^2 - bp_{12}x_1u + bp_{12}x_1h(x_1) + b(a - p_{12})x_2^2 \\ &= \left(u - \tfrac{1}{2}bp_{12}x_1\right)^2 - \tfrac{1}{4}b^2p_{12}^2x_1^2 + bp_{12}x_1h(x_1) + b(a - p_{12})x_2^2 \\ &\geq bp_{12}\left(\alpha_1 - \tfrac{1}{4}bp_{12}\right)x_1^2 + b(a - p_{12})x_2^2 \end{aligned}$$

Taking $p_{12} = ak$, where $0 < k < \min\{1, 4\alpha_1/(ab)\}$, ensures that p_{11}, $p_{11}p_{22} - p_{12}^2$, $bp_{12}\left(\alpha_1 - \frac{1}{4}bp_{12}\right)$, and $b(a - p_{12})$ are positive. Thus, the preceding inequality shows that the system is strictly passive. △

Example 5.5 Consider the pendulum equation

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu$$

where $b \geq 0$ and $c > 0$. View $y = x_2$ as the output and use the total energy

$$V(x) = \alpha[(1 - \cos x_1) + \tfrac{1}{2}x_2^2]$$

where $\alpha > 0$, as a storage function candidate. Note that when viewed as a function on the whole space R^2, $V(x)$ is positive semidefinite but not positive definite because it is zero at points other than the origin. We have

$$uy - \dot{V} = ux_2 - \alpha[x_2 \sin x_1 - x_2 \sin x_1 - bx_2^2 + cx_2u]$$

Taking $\alpha = 1/c$ to cancel the cross product term $x_2 u$, we obtain

$$uy - \dot{V} = (b/c)x_2^2 \geq 0$$

Hence, the system is passive when $b = 0$ and output strictly passive when $b > 0$. △

5.3 Positive Real Transfer Functions

Definition 5.4 *An $m \times m$ proper rational transfer function matrix $G(s)$ is positive real if*

- *poles of all elements of $G(s)$ are in $Re[s] \leq 0$,*
- *for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite, and*
- *any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian.*

It is strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$.

When $m = 1$, $G(j\omega) + G^T(-j\omega) = 2Re[G(j\omega)]$, an even function of ω. Therefore, the second condition of Definition 5.4 reduces to $Re[G(j\omega)] \geq 0$, $\forall\, \omega \in [0, \infty)$, which holds when the Nyquist plot of $G(j\omega)$ lies in the closed right-half complex plane. This is a condition that can be satisfied only if the relative degree of the transfer function is zero or one.[2]

The next lemma gives an equivalent characterization of strictly positive real transfer functions.[3]

Lemma 5.1 *Let $G(s)$ be an $m \times m$ proper rational transfer function matrix, and suppose* $\det[G(s) + G^T(-s)]$ *is not identically zero.*[4] *Then, $G(s)$ is strictly positive real if and only if*

- *$G(s)$ is Hurwitz; that is, poles of all elements of $G(s)$ have negative real parts,*
- *$G(j\omega) + G^T(-j\omega)$ is positive definite for all $\omega \in R$, and*
- *either $G(\infty) + G^T(\infty)$ is positive definite or it is positive semidefinite and $\lim_{\omega \to \infty} \omega^{2(m-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$, where $q = \mathrm{rank}[G(\infty) + G^T(\infty)]$.*

In the case $m = 1$, the frequency-domain condition of the lemma reduces to $Re[G(j\omega)] > 0$ for all $\omega \in [0, \infty)$ and either $G(\infty) > 0$ or $G(\infty) = 0$ and $\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] > 0$.

[2]The relative degree of a rational transfer function $G(s) = n(s)/d(s)$ is deg d - deg n. For a proper transfer function, the relative degree is a nonnegative integer.

[3]The proof is given in [30].

[4]Equivalently, $G(s) + G^T(-s)$ has a normal rank m over the field of rational functions of s.

Example 5.6 The transfer function $G(s) = 1/s$ is positive real since it has no poles in $Re[s] > 0$, has a simple pole at $s = 0$ whose residue is 1, and $Re[G(j\omega)] = 0$, $\forall\ \omega \neq 0$. It is not strictly positive real since $1/(s - \varepsilon)$ has a pole in $Re[s] > 0$ for any $\varepsilon > 0$. The transfer function $G(s) = 1/(s + a)$ with $a > 0$ is positive real, since it has no poles in $Re[s] \geq 0$ and $Re[G(j\omega)] = a/(\omega^2 + a^2) > 0$, $\forall\ \omega \in [0, \infty)$ Since this is so for every $a > 0$, we see that for any $\varepsilon \in (0, a)$ the transfer function $G(s - \varepsilon) = 1/(s + a - \varepsilon)$ will be positive real. Hence, $G(s) = 1/(s + a)$ is strictly positive real. The same conclusion can be drawn from Lemma 5.1 by noting that

$$\lim_{\omega\to\infty} \omega^2 Re[G(j\omega)] = \lim_{\omega\to\infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0$$

The transfer function $G(s) = 1/(s^2 + s + 1)$ is not positive real because its relative degree is two. We can see it also by calculating

$$Re[G(j\omega)] = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2} < 0, \quad \forall\ \omega > 1$$

Consider the 2×2 transfer function matrix

$$G(s) = \frac{1}{s+1} \begin{bmatrix} s+1 & 1 \\ -1 & 2s+1 \end{bmatrix}$$

Since $\det[G(s) + G^T(-s)]$ is not identically zero, we can apply Lemma 5.1. Noting that $G(\infty) + G^T(\infty)$ is positive definite and

$$G(j\omega) + G^T(-j\omega) = \frac{2}{\omega^2 + 1} \begin{bmatrix} \omega^2 + 1 & -j\omega \\ j\omega & 2\omega^2 + 1 \end{bmatrix}$$

is positive definite for all $\omega \in R$, we can conclude that $G(s)$ is strictly positive real. Finally, the 2×2 transfer function matrix

$$G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix} \quad \text{has} \quad G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

It can be verified that

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix}$$

is positive definite for all $\omega \in R$ and $\lim_{\omega\to\infty} \omega^2 \det[G(j\omega) + G^T(-j\omega)] = 4$. Consequently, by Lemma 5.1, we conclude that $G(s)$ is strictly positive real. △

Passivity properties of positive real transfer functions can be shown by using the next two lemmas, which are known, respectively, as the *positive real lemma* and the