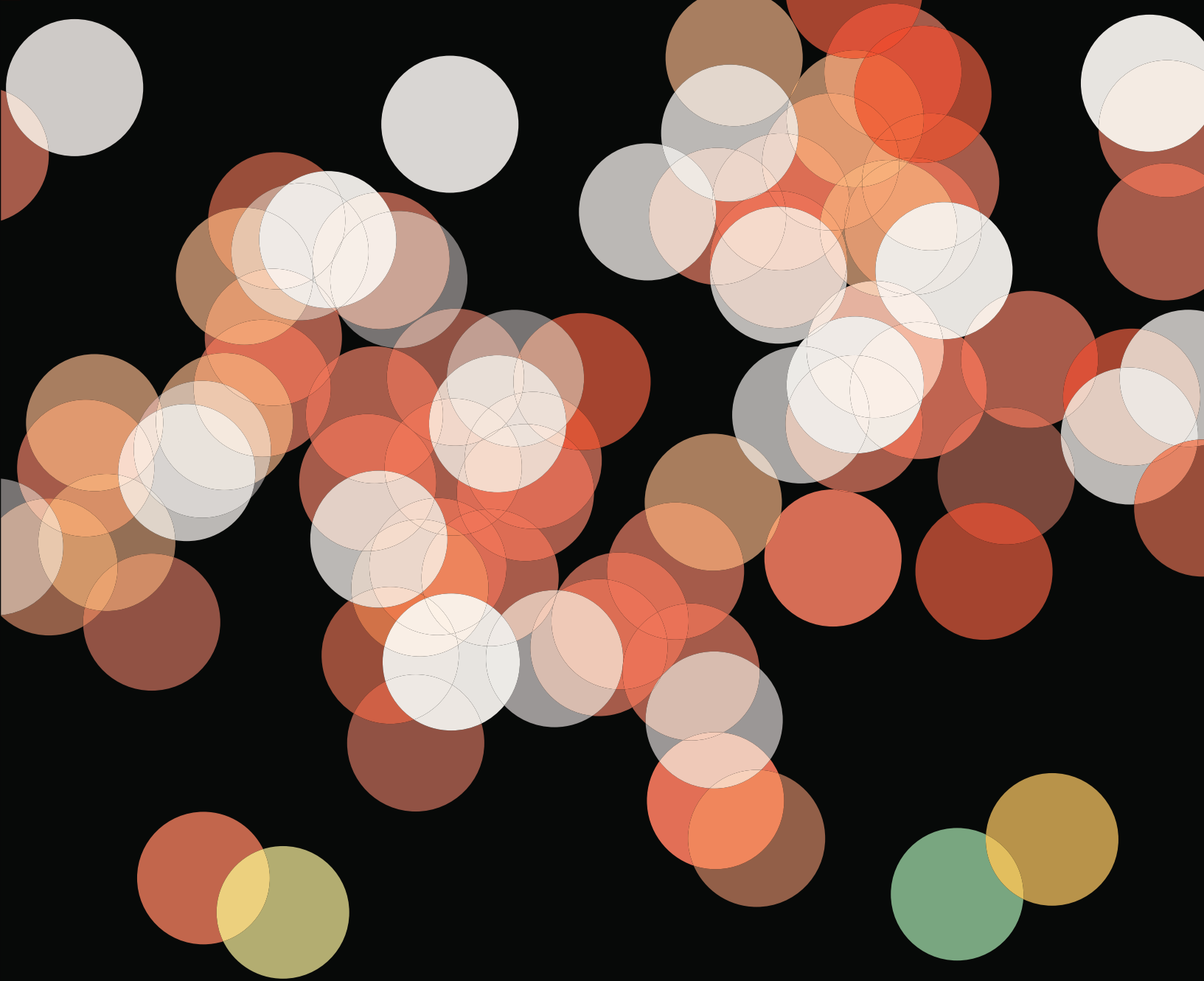
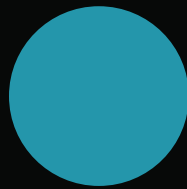




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Introduction to Analysis

William R. Wade
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- c) $\sum_{k=2}^{\infty} \log \left(\frac{k(k+2)}{(k+1)^2} \right)$
 d) $\sum_{k=1}^{\infty} 2 \sin \left(\frac{1}{k} - \frac{1}{k+1} \right) \cos \left(\frac{1}{k} + \frac{1}{k+1} \right)$

6.1.3. Prove that each of the following series diverges.

- a) $\sum_{k=1}^{\infty} \cos \left(\frac{1}{k^2} \right)$
 b) $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k} \right)^k$
 c) $\sum_{k=1}^{\infty} \frac{k+1}{k^2}$

6.1.4. Let $a_0, a_1, \dots$ be a sequence of real numbers. If $a_k \rightarrow L$ as $k \rightarrow \infty$, does

$$\sum_{k=1}^{\infty} (a_{k+1} - 2a_k + a_{k-1})$$

converge? If so, what is its value?

6.1.5. Find all $x \in \mathbf{R}$ for which

$$\sum_{k=1}^{\infty} (x^k - x^{k-1})(x^k + x^{k-1})$$

converges. For each such x , find the value of this series.

- 6.1.6.** a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then its partial sums s_n are bounded.
 b) Show that the converse of part a) is false. Namely, show that a series $\sum_{k=1}^{\infty} a_k$ may have bounded partial sums and still diverge.
- 6.1.7.** Suppose that I is a closed interval and $x_0 \in I$. Suppose further that f is differentiable on $\mathbf{R}$, that $f'(a) \neq 0$ for some $a \in \mathbf{R}$, that the function

$$F(x) := x - \frac{f(x)}{f'(a)} \quad x \in \mathbf{R}$$

satisfies $F(I) \subseteq I$, and that there is a number $0 < r < 1$ such that $f'(x)/f'(a) \in [1-r, 1]$ for all $x \in I$.

- a) Prove that $|F(x) - F(y)| \leq r|x - y|$ for all $x, y \in I$.
 b) If $x_n := F(x_{n-1})$ for $n \in \mathbf{N}$, prove that $|x_{n+1} - x_n| \leq r^n |x_1 - x_0|$ for all $n \in \mathbf{N}$.
 c) If $x_n = x_{n-1} - f(x_{n-1})/f'(a)$ for $n \in \mathbf{N}$, prove that

$$b := \lim_{n \rightarrow \infty} x_n$$

exists, belongs to I , and is a root of f ; that is, that $f(b) = 0$.

- 6.1.8.** a) Suppose that $\{a_k\}$ is a decreasing sequence of real numbers. Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $ka_k \rightarrow 0$ as $k \rightarrow \infty$.
 b) Let $s_n = \sum_{k=1}^n (-1)^{k+1}/k$ for $n \in \mathbf{N}$. Prove that s_{2n} is strictly increasing, s_{2n+1} is strictly decreasing, and $s_{2n+1} - s_{2n} \rightarrow 0$ as $n \rightarrow \infty$.
 c) Prove that part a) is false if *decreasing* is removed.

6.1.9. Let $\{b_k\}$ be a real sequence and $b \in \mathbf{R}$.

- a) Suppose that there are $M, N \in \mathbf{N}$ such that $|b - b_k| \leq M$ for all $k \geq N$. Prove that

$$\left| nb - \sum_{k=1}^n b_k \right| \leq \sum_{k=1}^N |b_k - b| + M(n - N)$$

for all $n > N$.

- b) Prove that if $b_k \rightarrow b$ as $k \rightarrow \infty$, then

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \rightarrow b$$

as $n \rightarrow \infty$.

- c) Show that the converse of b) is false.

6.1.10. A series $\sum_{k=0}^{\infty} a_k$ is said to be *Cesàro summable* to an $L \in \mathbf{R}$ if and only if

$$\sigma_n := \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) a_k$$

converges to L as $n \rightarrow \infty$.

- a) Let $s_n = \sum_{k=0}^{n-1} a_k$. Prove that

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}$$

for each $n \in \mathbf{N}$.

- b) Prove that if $a_k \in \mathbf{R}$ and $\sum_{k=0}^{\infty} a_k = L$ converges, then $\sum_{k=0}^{\infty} a_k$ is Cesàro summable to L .
 c) Prove that $\sum_{k=0}^{\infty} (-1)^k$ is Cesàro summable to $1/2$; hence the converse of b) is false.
 d) [TAUBER'S THEOREM]. Prove that if $a_k \geq 0$ for $k \in \mathbf{N}$ and $\sum_{k=0}^{\infty} a_k$ is Cesàro summable to L , then $\sum_{k=0}^{\infty} a_k = L$.

6.1.11. Suppose that $a_k \geq 0$ for k large and that $\sum_{k=1}^{\infty} a_k/k$ converges. Prove that

$$\lim_{j \rightarrow \infty} \sum_{k=1}^{\infty} \frac{a_k}{j+k} = 0.$$

6.1.12. If $\sum_{k=1}^n ka_k = (n+1)/(n+2)$ for $n \in \mathbf{N}$, prove that

$$\sum_{k=1}^{\infty} a_k = \frac{3}{4}.$$

6.2 SERIES WITH NONNEGATIVE TERMS

Although we obtained exact values in the preceding section for telescopic series and geometric series, finding exact values of a given series is frequently difficult, if not impossible. Fortunately, for many applications it is not as important to be able to find the value of a series as it is to know that the series converges. When it does converge, we can use its partial sums to approximate its value as accurately as we wish (up to the limitations of whatever computing device we are using). Therefore, much of this chapter is devoted to establishing tests which can be used to decide whether a given series converges or whether it diverges.

The partial sums of a divergent series may be bounded [like $\sum_{k=1}^{\infty} (-1)^k$] or unbounded [like $\sum_{k=1}^{\infty} 1/k$]. When the terms of a divergent series are nonnegative, the former cannot happen.

6.11 Theorem. Suppose that $a_k \geq 0$ for large k . Then $\sum_{k=1}^{\infty} a_k$ converges if and only if its sequence of partial sums $\{s_n\}$ is bounded; that is, if and only if there exists a finite number $M > 0$ such that

$$\left| \sum_{k=1}^n a_k \right| \leq M \text{ for all } n \in \mathbf{N}.$$

Proof. Set $s_n = \sum_{k=1}^n a_k$ for $n \in \mathbf{N}$. If $\sum_{k=1}^{\infty} a_k$ converges, then s_n converges as $n \rightarrow \infty$. Since every convergent sequence is bounded (Theorem 2.8), $\sum_{k=1}^{\infty} a_k$ has bounded partial sums.

Conversely, suppose that $|s_n| \leq M$ for $n \in \mathbf{N}$. Recall from Section 2.1 that $a_k \geq 0$ for large k means that there is an $N \in \mathbf{N}$ such that $a_k \geq 0$ for $k \geq N$. It follows that s_n is an increasing sequence when $n \geq N$. Hence by the Monotone Convergence Theorem (Theorem 2.19), s_n converges. ■

If $a_k \geq 0$ for large k , we shall write $\sum_{k=1}^{\infty} a_k < \infty$ when the series is convergent and $\sum_{k=1}^{\infty} a_k = \infty$ when the series is divergent.

In some cases, integration can be used to test convergence of a series. The idea behind this test is that

$$\int_1^{\infty} f(x) dx = \sum_{k=1}^{\infty} \int_k^{k+1} f(x) dx \approx \sum_{k=1}^{\infty} f(k)$$

when f is almost constant on each interval $[k, k+1]$. This will surely be the case for large k if $f(k) \downarrow 0$ as $k \rightarrow \infty$ (see Figure 6.1). This observation leads us to the following result.

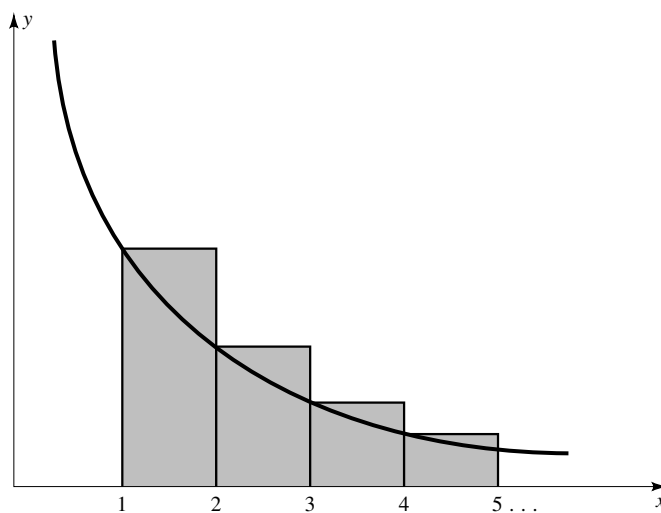


FIGURE 6.1

6.12 Theorem. [INTEGRAL TEST].

Suppose that $f : [1, \infty) \rightarrow \mathbf{R}$ is positive and decreasing on $[1, \infty)$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if f is improperly integrable on $[1, \infty)$; that is, if and only if

$$\int_1^{\infty} f(x) dx < \infty.$$

Proof. Let $s_n = \sum_{k=1}^n f(k)$ and $t_n = \int_1^n f(x) dx$ for $n \in \mathbf{N}$. Since f is decreasing, f is locally integrable on $[1, \infty)$ (see Exercise 5.1.8) and $f(k+1) \leq f(x) \leq f(k)$ for all $x \in [k, k+1]$. Hence, by the Comparison Theorem for Integrals,

$$f(k+1) \leq \int_k^{k+1} f(x) dx \leq f(k)$$

for $k \in \mathbf{N}$. Summing over $k = 1, \dots, n-1$, we obtain

$$s_n - f(1) = \sum_{k=2}^n f(k) \leq \int_1^n f(x) dx = t_n \leq \sum_{k=1}^{n-1} f(k) = s_n - f(n)$$

for all $n \geq N$. In particular,

$$f(n) \leq \sum_{k=1}^n f(k) - \int_1^n f(x) dx \leq f(1) \text{ for } n \in \mathbf{N}. \quad (3)$$

By (3) it is clear that $\{s_n\}$ is bounded if and only if $\{t_n\}$ is. Since $f(x) \geq 0$ implies that both s_n and t_n are increasing sequences, it follows from the Monotone Convergence Theorem that s_n converges if and only if t_n converges, as $n \rightarrow \infty$. ■

This test works best on series for which the integral of f can be easily computed or estimated. For example, to find out whether $\sum_{k=1}^{\infty} 1/(1+k^2)$ converges or diverges, let $f(x) = 1/(1+x^2)$ and observe that f is positive on $[1, \infty)$. Since $f'(x) = -2x/(1+x^2)^2$ is negative on $[1, \infty)$, it is also clear that f is decreasing. Since

$$\int_1^{\infty} \frac{dx}{1+x^2} = \arctan x \Big|_1^{\infty} = \frac{\pi}{2} - \arctan(1) < \infty,$$

it follows from the Integral Test that $\sum_{k=1}^{\infty} 1/(1+k^2)$ converges.

The Integral Test is most widely used in the following special case.

6.13 Corollary. [p-SERIES TEST]. *The series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \tag{4}$$

converges if and only if $p > 1$.

Proof. If $p = 1$ or $p \leq 0$, the series diverges. If $p > 0$ and $p \neq 1$, set $f(x) = x^{-p}$ and observe that $f'(x) = -px^{-p-1} < 0$ for all $x \in [1, \infty)$. Hence, f is nonnegative and decreasing on $[1, \infty)$. Since

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{n \rightarrow \infty} \frac{x^{1-p}}{1-p} \Big|_1^n = \lim_{n \rightarrow \infty} \frac{n^{1-p} - 1}{1-p}$$

has a finite limit if and only if $1-p < 0$, it follows from the Integral Test that (4) converges if and only if $p > 1$. ■

The Integral Test, which requires f to satisfy some very restrictive hypotheses, has limited applications. The following test can be used in a much broader context.

6.14 Theorem. [COMPARISON TEST].

Suppose that $0 \leq a_k \leq b_k$ for large k .

- i) *If $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.*
- ii) *If $\sum_{k=1}^{\infty} a_k = \infty$, then $\sum_{k=1}^{\infty} b_k = \infty$.*

Proof. By hypothesis, choose $N \in \mathbf{N}$ so large that $0 \leq a_k \leq b_k$ for $k > N$. Set $s_n = \sum_{k=1}^n a_k$ and $t_n = \sum_{k=1}^n b_k$, $n \in \mathbf{N}$. Then $0 \leq s_n - s_N \leq t_n - t_N$ for all $n \geq N$. Since N is fixed, it follows that s_n is bounded when t_n is, and t_n is unbounded when s_n is. Apply Theorem 6.11 and the proof of the theorem is complete. ■

The Comparison Test is used to compare one series with another whose convergence property is already known (e.g., a p -series or a geometric series). Frequently, the inequalities $|\sin x| \leq |x|$ for all $x \in \mathbf{R}$ (see Appendix B) and $|\log x| \leq x^\alpha$ for each $\alpha > 0$ provided x is sufficiently large (see Exercise 4.4.6) are helpful in this regard. Although there is no simple algorithm for this process, the idea is to examine the terms of the given series, ignoring the superfluous factors, and dominating the more complicated factors by simpler ones. Here is a typical example.

6.15 EXAMPLE.

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{3k}{k^2 + k} \sqrt{\frac{\log k}{k}} \quad (5)$$

converges or diverges.

Solution. The k th term of this series can be written by using three factors:

$$\frac{1}{k} \frac{3k}{k+1} \sqrt{\frac{\log k}{k}}.$$

The factor $3k/(k+1)$ is dominated by 3. Since $\log k \leq \sqrt{k}$ for large k , the factor $\sqrt{\log k/k}$ satisfies

$$\sqrt{\frac{\log k}{k}} \leq \sqrt{\frac{\sqrt{k}}{k}} = \frac{1}{\sqrt[4]{k}}$$

for large k . Therefore, the terms of (5) are dominated by $3/k^{5/4}$. Since $\sum_{k=1}^{\infty} 3/k^{5/4}$ converges by the p -Series Test, it follows from the Comparison Test that (5) converges. ■

The Comparison Test may not be easy to apply to a given series, even when we know which series it should be compared with, because the process of comparison often involves use of delicate inequalities. For situations like this, the following test is usually more efficient.