PEARSON NEW INTERNATIONAL EDITION

Linear Algebra & Differential Equations

Gary L. Peterson James S. Sochacki

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and multiplying by it gives us

$$\frac{d}{dx}(x^2 y) = 3x^2.$$

Integrating and solving for y, we obtain

$$y = x + Cx^{-2}.$$

Using the initial condition, we find $C = -1$ and consequently the solution to the initial value problem is

$$y = x - x^{-2}.$$

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EXAMPLE 5 Solve the initial value problem

$$y' \cos x + xy \cos x = \cos x, \qquad y(0) = 1.$$

Solution Upon dividing by $\cos x$, we obtain

$$y' + xy = 1.$$

We find that an integrating factor is

$$e^{\int x\,dx} = e^{x^2/2}.$$

Multiplying the differential equation by it gives us

$$\frac{d}{dx}(e^{x^2/2} y) = e^{x^2/2}.$$

Since there is no closed form for an antiderivative of $e^{x^2/2}$, we will use an antiderivative of the form $F(x) = \int_a^x f(t)\,dt$ for $e^{x^2/2}$:

$$\int_0^x e^{t^2/2}\,dt.$$

We now have

$$\frac{d}{dx}(e^{x^2/2} y) = \frac{d}{dx}\int_0^x e^{t^2/2}\,dt,$$

so that

$$e^{x^2/2} y = \int_0^x e^{t^2/2}\,dt + C.$$

Using the initial condition $y(0) = 1$ gives us

$$y(0) = \int_0^0 e^{t^2/2}\,dt + C = 0 + C = C = 1.$$

Therefore we have

$$e^{x^2/2}y = \int_0^x e^{t^2/2}\,dt + 1,$$

or

$$y = e^{-x^2/2}\int_0^x e^{t^2/2}\,dt + e^{-x^2/2}.$$

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We can use Maple to graph the solution to Example 5 by typing and entering

```
plot(exp(-x^2/2)*int(exp(t^2/2),t=0..x)+exp(-x^2/2),x=0..5);
```

This graph appears in Figure 3.8.

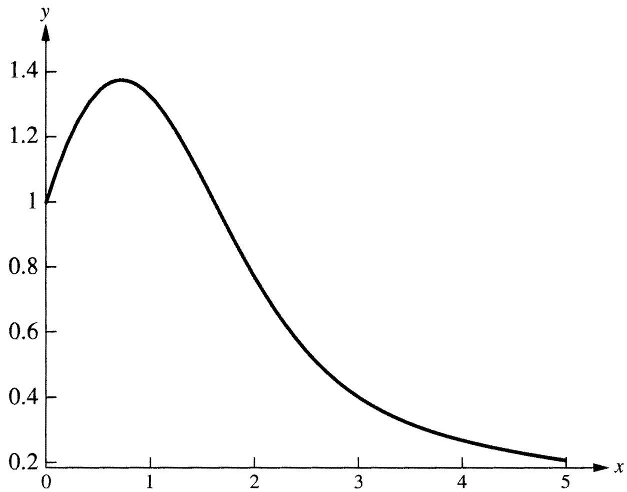

Figure 3.8

EXERCISES 3.4

In Exercises 1–12, determine the integrating factor and solve the differential equation.

1. $y' + y/x^2 = 0$

2. $y' = y/x - 2,\ x > 0$

3. $y' - 2xy = x$

4. $y' = 4y + 2x$

5. $y' = 1 + \dfrac{y}{1+2x},\ x > 0$

6. $y' = x^2 - \dfrac{2xy}{1+x^2}$

7. $xy' + y = x^2,\ x > 0$

8. $xy' + y = x^2,\ x < 0$

9. $(1+xy)\,dx - x^2\,dy = 0,\ x < 0$

10. $(1+xy)\,dx - x^2\,dy = 0,\ x > 0$

11. $\dfrac{dy}{dt} + e^t y = e^t$

12. $\dfrac{dr}{d\theta} = r\tan\theta + \sin\theta,\ 0 < \theta < \pi/2$

In Exercises 13–18, solve the initial value problems.

13. $y' + 4y = 2,\ y(1) = 2$

14. $2xy' + y = 1,\ y(4) = 0$

15. $y' + \dfrac{y}{x+1} = 2,\ y(0) = 2$

16. $y' + 2xy = 1,\ y(0) = -1$

17. $y' = \cos 2x - y/x,\ y(\pi/2) = 0$

18. $x(x+1)y' = 2 + y,\ y(1) = 0$

19. Use Maple (or another appropriate software package) to graph the solution in Exercise 13.

20. Use Maple (or another appropriate software package) to graph the solution in Exercise 14.

21. Show that the solution to the linear first order initial value problem

$$y' + p(x)y = q(x), \qquad y(x_0) = y_0$$

is given by

$$y(x) = e^{-\int_{x_0}^{x} p(t)\,dt} \int_{x_0}^{x} u(t)q(t)\,dt + y_0 e^{-\int_{x_0}^{x} p(t)\,dt}$$

where $u(t) = e^{\int_{x_0}^{t} p(s)\,ds}$.

22. Use the result of Exercise 21 to do Exercise 13.

3.5 More Techniques for Solving First Order Differential Equations

In the previous three sections we considered three types of first order differential equations: separable equations, exact equations, and linear equations. If a first order equation is not of one of these types, there are methods that sometimes can be used to convert the equation to one of these three types, which then gives us a way of solving the differential equation. In this section we will give a sampling of some of these techniques.

The first approach we consider involves attempting to convert a differential equation

$$\boxed{r(x, y)\,dx + s(x, y)\,dy = 0} \tag{1}$$

that is not exact into an exact one. The idea is to try to find a function I of x and y so that when we multiply our differential equation in (1) by $I(x, y)$ obtaining

$$\boxed{I(x, y)r(x, y)\,dx + I(x, y)s(x, y)\,dy = 0,} \tag{2}$$

we have an exact differential equation.

In the previous section we multiplied the linear differential equation

$$y' + p(x)y = q(x)$$

by $u = e^{\int p(x)\,dx}$ because it made the left-hand side of the linear differential equation $d/dx(uy)$. Since we could now integrate this derivative, we called u an integrating factor. In the same spirit, we will refer to $I(x, y)$ as an **integrating factor** for the differential equation in (1) if it converts Equation (1) into Equation (2) with a left-hand side that is a total differential. (Actually, multiplying the linear equation $y' + p(x)y = q(x)$ by u does make the differential equation into an exact one—see Exercise 7. Thus our old meaning of integrating factor is a special case of its new meaning.)

If we let

$$M(x, y) = I(x, y)r(x, y)$$

and

$$N(x, y) = I(x, y)s(x, y)$$

in Theorem 3.2, for exactness we must have

$$\frac{\partial}{\partial y} I(x, y)r(x, y) = \frac{\partial}{\partial x} I(x, y)s(x, y). \tag{3}$$

Numerous techniques have been devised in attempts to find integrating factors that satisfy the above condition. One ploy is to try to find an integrating factor of the form

$$\boxed{I(x, y) = x^m y^n}$$

where m and n are constants. The following example illustrates this approach.

EXAMPLE 1 Solve the differential equation

$$(x^2y + y^2)\,dx + (x^3 + 2xy)\,dy = 0.$$

Solution It is easily seen that this equation is not exact. Let us see if we can find an integrating factor of the form $I(x, y) = x^m y^n$. Multiplying the differential equation by this, we have the differential equation

$$x^m y^n[(x^2y + y^2)\,dx + (x^3 + 2xy)\,dy] = 0$$

or

$$(x^{m+2}y^{n+1} + x^m y^{n+2})\,dx + (x^{m+3}y^n + 2x^{m+1}y^{n+1})\,dy = 0. \tag{4}$$

With

$$M(x, y) = x^{m+2}y^{n+1} + x^m y^{n+2} \quad \text{and} \quad N(x, y) = x^{m+3}y^n + 2x^{m+1}y^{n+1},$$

Equation (3) tells us that we must have

$$M_y(x, y) = (n + 1)x^{m+2}y^n + (n + 2)x^m y^{n+1}$$

the same as

$$N_x(x, y) = (m + 3)x^{m+2}y^n + 2(m + 1)x^m y^{n+1}$$

for Equation (4) to be exact. At this point we make the observation that $M_y(x, y)$ and $N_x(x, y)$ will be the same if the coefficients of the $x^{m+2}y^n$ terms are the same and the coefficients of the $x^m y^{n+1}$ terms are the same in these two expressions. This gives us the system of linear equations

$$\begin{aligned} n + 1 &= m + 3 \\ n + 2 &= 2(m + 1), \end{aligned}$$

which you can easily solve finding

$$m = 2 \quad \text{and} \quad n = 4.$$

Substituting these values of m and n into Equation (4), we have the exact equation

$$(x^4y^5 + x^2y^6)\,dx + (x^5y^4 + 2x^3y^5)\,dy = 0.$$

Let us now solve this exact equation. We have

$$F(x, y) = \int (x^4y^5 + x^2y^6)\,dx + h(y)$$
$$= \frac{x^5y^5}{5} + \frac{x^3y^6}{3} + h(y)$$

and

$$F_y(x, y) = x^5y^4 + 2x^3y^5 + h'(y) = x^5y^4 + 2x^3y^5$$

so that

$$h'(y) = 0$$

and

$$h(y) = 0.$$

Our solutions are then given by

$$\frac{x^5y^5}{5} + \frac{x^3y^6}{3} = C.$$

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Do not expect the technique of Example 1 to always work. Indeed, if you were to try to apply it to the equation

$$(x^2 + y^2 + 1)\,dx + (xy + y)\,dy = 0,$$

you would find that the resulting system of equations in n and m obtained by comparing coefficients as we did in Example 1 has no solution. (Try it.) There is, however, a way of finding an integrating factor for this equation. It involves looking for one that is a function of x only; that is, we look for an integrating factor of the form

$$\boxed{I(x, y) = f(x).}$$

Indeed, Equation (3) tells us the differential equation is exact if and only if

$$\frac{\partial}{\partial y}(f(x)r(x, y)) = \frac{\partial}{\partial x}(f(x)s(x, y))$$

or

$$f(x)r_y(x, y) = f(x)s_x(x, y) + f'(x)s(x, y).$$

Solving this equation for $f'(x)$, we have

$$f'(x) = \frac{r_y(x, y) - s_x(x, y)}{s(x, y)} f(x)$$

or, using u for $f(x)$,

$$\boxed{\frac{du}{dx} = \frac{r_y(x, y) - s_x(x, y)}{s(x, y)} u.} \tag{5}$$

If the expression

$$\boxed{\frac{r_y(x, y) - s_x(x, y)}{s(x, y)}} \tag{6}$$

is a function of x only, Equation (5) is a separable equation that can be solved for $u = f(x)$. In other words, we have just arrived at the following procedure for finding an integrating factor that is a function of x:

1. Calculate the expression in (6).
2. If the expression in (6) is a function of x only, solve the separable differential equation in (5) to obtain an integrating factor $u = f(x)$.
3. Use this integrating factor to solve the differential equation.

Of course, our procedure does not apply if the expression in (6) involves y. A similar approach can be used to attempt to obtain integrating factors that are functions of y only. See Exercise 8 for the details.

Let us use the procedure we have just obtained to solve a differential equation.

EXAMPLE 2 Solve the differential equation

$$(x^2 + y^2 + 1)\, dx + (xy + y)\, dy = 0.$$

Solution With

$$r(x, y) = x^2 + y^2 + 1 \quad \text{and} \quad s(x, y) = xy + y,$$

we have

$$\frac{r_y(x, y) - s_x(x, y)}{s(x, y)} = \frac{2y - y}{xy + y} = \frac{1}{x + 1}$$

is a function of x only. Now we solve the separable equation

$$\frac{du}{dx} = \frac{1}{x + 1} \cdot u$$

for the integrating factor u. This gives

$$\frac{du}{u} = \frac{1}{x + 1}\, dx$$

$$\ln u = \ln(x + 1)$$

or

$$u = x + 1.$$