Nonlinear Systems

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Thus, $V(t,x)$ satisfies the first inequality of the theorem with

$$c_1 = \frac{(1 - e^{-2L\delta})}{2L} \quad \text{and} \quad c_2 = \frac{k^2(1 - e^{-2\lambda\delta})}{2\lambda}$$

To calculate the derivative of V along the trajectories of the system, define the sensitivity functions

$$\phi_t(\tau; t, x) = \frac{\partial}{\partial t}\phi(\tau; t, x); \quad \phi_x(\tau; t, x) = \frac{\partial}{\partial x}\phi(\tau; t, x)$$

Then,

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &= \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - \phi^T(t; t, x)\phi(t; t, x) \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_t(\tau; t, x)\ d\tau \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_x(\tau; t, x)\ d\tau f(t,x) \\
&= \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - \|x\|_2^2 \\
&\quad + \int_t^{t+\delta} 2\phi^T(\tau; t, x)[\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t,x)]\ d\tau
\end{aligned}$$

It is not difficult to show that[26]

$$\phi_t(\tau; t, x) + \phi_x(\tau; t, x) f(t,x) \equiv 0, \quad \forall\ \tau \geq t$$

Therefore,

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) &= \phi^T(t+\delta; t, x)\phi(t+\delta; t, x) - \|x\|_2^2 \\
&\leq -(1 - k^2 e^{-2\lambda\delta})\|x\|_2^2
\end{aligned}$$

By choosing $\delta = \ln(2k^2)/(2\lambda)$, the second inequality of the theorem is satisfied with $c_3 = 1/2$. To show the last inequality, let us note that $\phi_x(\tau; t, x)$ satisfies the sensitivity equation

$$\frac{\partial}{\partial \tau}\phi_x = \frac{\partial f}{\partial x}(\tau, \phi(\tau; t, x))\ \phi_x, \quad \phi_x(t; t, x) = I$$

Since

$$\left\| \frac{\partial f}{\partial x}(t,x) \right\|_2 \leq L$$

[26]See Exercise 3.30.

on D, ϕ_x satisfies the bound[27]

$$\|\phi_x(\tau; t, x)\|_2 \leq e^{L(\tau - t)}$$

Therefore,

$$\begin{aligned} \left\| \frac{\partial V}{\partial x} \right\|_2 &= \left\| \int_t^{t+\delta} 2\phi^T(\tau; t, x)\phi_x(\tau; t, x) \; d\tau \right\|_2 \\ &\leq \int_t^{t+\delta} 2\|\phi(\tau; t, x)\|_2 \; \|\phi_x(\tau; t, x)\|_2 \; d\tau \\ &\leq \int_t^{t+\delta} 2ke^{-\lambda(\tau - t)} \; e^{L(\tau - t)} \; d\tau \; \|x\|_2 \\ &= \frac{2k}{(\lambda - L)}[1 - e^{-(\lambda - L)\delta}]\|x\|_2 \end{aligned}$$

Thus, the last inequality of the theorem is satisfied with

$$c_4 = \frac{2k}{(\lambda - L)}[1 - e^{-(\lambda - L)\delta}]$$

If all the assumptions hold globally, then clearly r_0 can be chosen arbitrarily large. If the system is autonomous, then $\phi(\tau; t, x)$ depends only on $(\tau - t)$; that is,

$$\phi(\tau; t, x) = \psi(\tau - t; x)$$

Then,

$$V(t, x) = \int_t^{t+\delta} \psi^T(\tau - t; x)\psi(\tau - t; x) \; d\tau = \int_0^{\delta} \psi^T(s; x)\psi(s; x) \; ds$$

which is independent of t. □

In Theorem 4.13, we saw that if the linearization of a nonlinear system about the origin has an exponentially stable equilibrium point, then the origin is an exponentially stable equilibrium point for the nonlinear system. We will use Theorem 4.14 to prove that exponential stability of the linearization is a necessary and sufficient condition for exponential stability of the origin.

Theorem 4.15 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\|_2 < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \left. \frac{\partial f}{\partial x}(t, x) \right|_{x=0}$$

[27] See Exercise 3.17.

Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if it is an exponentially stable equilibrium point for the linear system

$$\dot{x} = A(t)x$$

◇

Proof: The "if" part follows from Theorem 4.13. To prove the "only if" part, write the linear system as

$$\dot{x} = f(t,x) - [f(t,x) - A(t)x] = f(t,x) - g(t,x)$$

Recalling the argument preceding Theorem 4.13, we know that

$$\|g(t,x)\|_2 \le L\|x\|_2^2, \quad \forall\ x \in D,\ \forall\ t \ge 0$$

Since the origin is an exponentially stable equilibrium of the nonlinear system, there are positive constants k, λ, and c such that

$$\|x(t)\|_2 \le k\|x(t_0)\|_2 e^{-\lambda(t-t_0)}, \quad \forall\ t \ge t_0 \ge 0,\ \forall\ \|x(t_0)\|_2 < c$$

Choosing $r_0 < \min\{c, r/k\}$, all the conditions of Theorem 4.14 are satisfied. Let $V(t,x)$ be the function provided by Theorem 4.14 and use it as a Lyapunov function candidate for the linear system. Then,

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}A(t)x &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x) - \frac{\partial V}{\partial x}g(t,x) \\ &\le -c_3\|x\|_2^2 + c_4 L\|x\|_2^3 \\ &< -(c_3 - c_4 L\rho)\|x\|_2^2, \quad \forall\ \|x\|_2 < \rho \end{aligned}$$

The choice $\rho < \min\{r_0, c_3/(c_4 L)\}$ ensures that $\dot{V}(t,x)$ is negative definite in $\|x\|_2 < \rho$. Consequently, all the conditions of Theorem 4.10 are satisfied in $\|x\|_2 < \rho$, and we conclude that the origin is an exponentially stable equilibrium point for the linear system. □

Corollary 4.3 *Let $x = 0$ be an equilibrium point of the nonlinear system $\dot{x} = f(x)$, where $f(x)$ is continuously differentiable in some neighborhood of $x = 0$. Let $A = [\partial f/\partial x](0)$. Then, $x = 0$ is an exponentially stable equilibrium point for the nonlinear system if and only if A is Hurwitz.* ◇

Example 4.23 Consider the first-order system $\dot{x} = -x^3$. We saw in Example 4.14 that the origin is asymptotically stable, but linearization about the origin results in the linear system $\dot{x} = 0$, whose A matrix is not Hurwitz. Using Corollary 4.3, we conclude that the origin is not exponentially stable. △

The following converse Lyapunov theorems (Theorem 4.16 and 4.17) extend Theorem 4.15 in two different directions, but their proofs are more involved. Theorem 4.16 applies to the more general case of uniform asymptotic stability.[28] Theorem 4.17 applies to autonomous systems and produces a Lyapunov function that is defined on the whole region of attraction.

Theorem 4.16 *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \mid \|x\| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let β be a class $\mathcal{KL}$ function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the trajectory of the system satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\ x(t_0) \in D_0,\ \forall\ t \ge t_0 \ge 0$$

Then, there is a continuously differentiable function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -\alpha_3(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le \alpha_4(\|x\|)$$

where α_1, α_2, α_3, and α_4 are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t. ◇

Proof: See Appendix C.7.

Theorem 4.17 *Let $x = 0$ be an asymptotically stable equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \to R^n$ is locally Lipschitz and $D \subset R^n$ is a domain that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \le -W(x), \quad \forall\ x \in R_A$$

and for any $c > 0$, $\{V(x) \le c\}$ is a compact subset of R_A. When $R_A = R^n$, $V(x)$ is radially unbounded. ◇

[28] Theorem 4.16 can be stated for a function $f(t, x)$ that is only locally Lipschitz, rather than continuously differentiable [125, Theorem 14]. It is also possible to state the theorem for the case of global uniform asymptotic stability [125, Theorem 23]

Proof: See Appendix C.8.

An interesting feature of Theorem 4.17 is that any bounded subset S of the region of attraction can be included in a compact set of the form $\{V(x) \le c\}$ for some constant $c > 0$. This feature is useful because quite often we have to limit our analysis to a positively invariant, compact set of the form $\{V(x) \le c\}$. With the property $S \subset \{V(x) \le c\}$, our analysis will be valid for the whole set S. If, on the other hand, all we know is the existence of a Lyapunov function $V_1(x)$ on S, we will have to choose a constant c_1 such that $\{V_1(x) \le c_1\}$ is compact and included in S; then our analysis will be limited to $\{V_1(x) \le c_1\}$, which is only a subset of S.

4.8 Boundedness and Ultimate Boundedness

Lyapunov analysis can be used to show boundedness of the solution of the state equation, even when there is no equilibrium point at the origin. To motivate the idea, consider the scalar equation

$$\dot{x} = -x + \delta \sin t, \quad x(t_0) = a, \quad a > \delta > 0$$

which has no equilibrium points and whose solution is given by

$$x(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin \tau \; d\tau$$

The solution satisfies the bound

$$\begin{aligned} |x(t)| &\le e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \; d\tau \;=\; e^{-(t-t_0)}a + \delta \left[1 - e^{-(t-t_0)}\right] \\ &\le a, \quad \forall \; t \ge t_0 \end{aligned}$$

which shows that the solution is bounded for all $t \ge t_0$, uniformly in t_0, that is, with a bound independent of t_0. While this bound is valid for all $t \ge t_0$, it becomes a conservative estimate of the solution as time progresses, because it does not take into consideration the exponentially decaying term. If, on the other hand, we pick any number b such that $\delta < b < a$, it can be easily seen that

$$|x(t)| \le b, \quad \forall \; t \ge t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$$

The bound b, which again is independent of t_0, gives a better estimate of the solution after a transient period has passed. In this case, the solution is said to be uniformly ultimately bounded and b is called the ultimate bound. Showing that the solution of $\dot{x} = -x + \delta \sin t$ has the uniform boundedness and ultimate boundedness properties can be done via Lyapunov analysis without using the explicit solution of the state equation. Starting with $V(x) = x^2/2$, we calculate the derivative of V along the trajectories of the system, to obtain

$$\dot{V} = x\dot{x} = -x^2 + x\delta \sin t \leq -x^2 + \delta|x|$$

The right-hand side of the foregoing inequality is not negative definite because, near the origin, the positive linear term $\delta|x|$ dominates the negative quadratic term $-x^2$. However, $\dot{V}$ is negative outside the set $\{|x| \leq \delta\}$. With $c > \delta^2/2$, solutions starting in the set $\{V(x) \leq c\}$ will remain therein for all future time since $\dot{V}$ is negative on the boundary $V = c$. Hence, the solutions are uniformly bounded. Moreover, if we pick any number ε such that $(\delta^2/2) < \varepsilon < c$, then $\dot{V}$ will be negative in the set $\{\varepsilon \leq V \leq c\}$, which shows that, in this set, V will decrease monotonically until the solution enters the set $\{V \leq \varepsilon\}$. From that time on, the solution cannot leave the set $\{V \leq \varepsilon\}$ because $\dot{V}$ is negative on the boundary $V = \varepsilon$. Thus, we can conclude that the solution is uniformly ultimately bounded with the ultimate bound $|x| \leq \sqrt{2\varepsilon}$.

The purpose of this section is to show how Lyapunov analysis can be used to draw similar conclusions for the system

$$\dot{x} = f(t, x) \tag{4.32}$$

where $f : [0, \infty) \times D \to R^n$ is piecewise continuous in t and locally Lipschitz in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain that contains the origin.

Definition 4.6 *The solutions of* (4.32) *are*

- *uniformly bounded if there exists a positive constant c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $\beta = \beta(a) > 0$, independent of t_0, such that*

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq \beta, \quad \forall\, t \geq t_0 \tag{4.33}$$

- *globally uniformly bounded if* (4.33) *holds for arbitrarily large a.*

- *uniformly ultimately bounded with ultimate bound b if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0, such that*

$$\|x(t_0)\| \leq a \;\Rightarrow\; \|x(t)\| \leq b, \quad \forall\, t \geq t_0 + T \tag{4.34}$$

- *globally uniformly ultimately bounded if* (4.34) *holds for arbitrarily large a.*

In the case of autonomous systems, we may drop the word "uniformly" since the solution depends only on $t - t_0$.

To see how Lyapunov analysis can be used to study boundedness and ultimate boundedness, consider a continuously differentiable, positive definite function $V(x)$ and suppose that the set $\{V(x) \leq c\}$ is compact, for some $c > 0$. Let

$$\Lambda = \{\varepsilon \leq V(x) \leq c\}$$