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and Abū Kāmil proceeded to solve for x by first squaring both sides of this equation. The final result is

$$x = 10 + \sqrt{50} - \sqrt{50 + \sqrt{20{,}000} - \sqrt{5{,}000}}.$$

Abū Kāmil could even solve systems of equations. Consider problem 61: "One says that 10 is divided into three parts, and if the smallest is multiplied by itself and added to the middle one multiplied by itself, it equals the largest multiplied by itself, and when the smallest is multiplied by the largest, it equals the middle multiplied by itself."[21] In modern symbols, we are asked to find $x < y < z$, where

$$x + y + z = 10, \qquad x^2 + y^2 = z^2, \quad \text{and} \quad xz = y^2.$$

Presumably noticing that the three equations are all homogeneous, Abū Kāmil used the ancient method of false position. Namely, he initially ignored the first equation and set $x = 1$ in the second and third equations to get $1 + y^2 = y^4$. Since this is an equation in quadratic form, he could solve it:

$$z = y^2 = \frac{1}{2} + \sqrt{\frac{5}{4}} \quad \text{and} \quad y = \sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}.$$

Now, returning to the first equation, he noted that the sum of his three "false" values was

$$1\frac{1}{2} + \sqrt{\frac{5}{4}} + \sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}},$$

instead of 10. To find the correct values, he needed to divide 10 by this value and multiply the quotient by the "false" values. Since the false value of x was 1, this just meant that the correct value for x was

$$x = \frac{10}{1\frac{1}{2} + \sqrt{\frac{5}{4}} + \sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}}.$$

To simplify this was not a trivial procedure, but Abū Kāmil began by multiplying the denominator by x and setting the product equal to 10. He ultimately turned this equation into a quadratic equation and succeeded in determining that

$$x = 5 - \sqrt{\sqrt{3{,}125} - 50}.$$

To find y and z by multiplying the false values by this quotient would have been even more difficult, so he chose to find z by beginning the problem anew with the false value $z = 1$. Of course, once he found z, he could determine y by subtraction.

When considering Abū Kāmil's algebra, remember that, like all Islamic algebra texts of his era, it was written without symbols. Thus, the algebraic manipulation that modern symbolism makes almost obvious is carried out completely verbally. (Of course, in our final example, the procedure is by no means "obvious," even with symbolism.) More importantly, however, Abū Kāmil was willing to use the algebraic algorithms that had been systematized by the time of al-Khwārizmī with any type of positive "number." He made no distinction between operating with 2 or with $\sqrt{8}$ or even with $\sqrt{\sqrt{2}-1}$. Since these algorithms came from geometry, on one level that is not surprising. After all, it was the Greek failure to find a "numerical" representation of the diagonal of a square that was one of the reasons for their use of the geometric algebra of line segments and areas. But in dealing with these quantities, Abū Kāmil interpreted all of them in the same way. It did not matter whether a magnitude was technically a square or a fourth power or a root or a root of a root. For Abū Kāmil, the solution of a quadratic equation was not a line segment, as it would be in the interpretation of the appropriate propositions of the *Elements*. It was a "number," even though Abū Kāmil could not perhaps give a proper definition of that term. He therefore had no compunction about combining the various quantities that appeared in the solutions, using general rules. Abū Kāmil's willingness to handle all of these quantities by the same techniques helped pave the way toward a new understanding of the concept of number that was just as important as al-Samaw'al's use of decimal approximations.

9.3.3 Al-Karajī, al-Samaw'al, and the Algebra of Polynomials

The process of relating arithmetic to algebra, begun by al-Khwārizmī and Abū Kāmil, continued in the Islamic world with the work of Abū Bakr al-Karajī (d. 1019) and al-Samaw'al over the next two centuries. These latter mathematicians were instrumental in showing that the techniques of arithmetic could be fruitfully applied in algebra and, reciprocally, that ideas originally developed in algebra could also be important in dealing with numbers.

Little is known of the life of al-Karajī other than that he worked in Baghdad around the year 1000 and wrote many mathematics works as well as works on engineering topics. In the first decade of the eleventh century, he composed a major work on algebra entitled *al-Fakhrī* (*The Marvelous*). The aim of *al-Fakhrī*, and of algebra in general according to al-Karajī, was "the determination of unknowns starting from knowns."[22] In pursuit of this aim, he made use of all the techniques of arithmetic, converted into techniques of dealing with unknowns. He began by making a systematic study of the algebra of exponents. Although earlier writers, including Diophantus, had considered powers of the unknown greater than the third, al-Karajī was the first to fully understand that these powers can be extended indefinitely. In fact, he developed a method of naming the various powers x^n and their reciprocals $\frac{1}{x^n}$. Each power was defined recursively as x times the previous power. It followed that there was an infinite sequence of proportions,

$$1 : x = x : x^2 = x^2 : x^3 = \ldots,$$

and a similar one for reciprocals,

$$\frac{1}{x} : \frac{1}{x^2} = \frac{1}{x^2} : \frac{1}{x^3} = \frac{1}{x^3} : \frac{1}{x^4} = \ldots.$$

Once the powers were understood, al-Karajī could establish general procedures for adding, subtracting, and multiplying monomials and polynomials. In division, however, he only used monomials as divisors, partly because he was unable to incorporate rules for negative numbers into his theory and partly because of his verbal means of expression. Similarly, although he developed an algorithm for calculating square roots of polynomials, it was only applicable in limited circumstances.

BIOGRAPHY

Al-Samaw'al (c. 1125–1174)

Al-Samaw'al was born in Baghdad to well-educated Jewish parents. His father was in fact a Hebrew poet. Besides giving him a religious education, they encouraged him to study medicine and mathematics. Because the House of Wisdom no longer existed in Baghdad, he had to study mathematics independently and therefore traveled to various other parts of the Middle East. He wrote his major mathematical work, *Al-Bāhir*, when he was only nineteen. His interests later turned to medicine, and he became a successful physician and author of medical texts. The only extant one is entitled *The Companion's Promenade in the Garden of Love*, a treatise on sexology and a collection of erotic stories. When he was about forty, he decided to convert to Islam. To justify his conversion to the world, he wrote an autobiography in 1167 stating his arguments against Judaism, a work that became famous as a source of Islamic polemics against the Jews.

Al-Karajī was more successful in continuing the work of Abū Kāmil in applying arithmetic operations to irrational quantities. In particular, he explicitly interpreted the various classes of incommensurables in *Elements* X as classes of "numbers" on which the various operations of arithmetic were defined, but then noted that there were indefinitely many other classes composed of three or more surds. Like Abū Kāmil, he gave no definition of "number," but just dealt with the various surd quantities using numerical rather than geometrical techniques. As part of this process, he developed various formulas involving surds, such as

$$\sqrt{A+B} = \sqrt{\frac{A+\sqrt{A^2-B^2}}{2}} + \sqrt{\frac{A-\sqrt{A^2-B^2}}{2}}$$

and

$$\sqrt[3]{A} + \sqrt[3]{B} = \sqrt[3]{3\sqrt[3]{A^2B} + 3\sqrt[3]{AB^2} + A + B}.$$

Further work in dealing with algebraic manipulation was accomplished by al-Samaw'al, who, in particular, introduced negative coefficients. He expressed his rules for dealing with these coefficients quite clearly in his algebra text *Al-Bāhir fi'l-ḥisāb* (*The Shining Book of Calculation*):

> If we subtract an additive number from an empty power $[0x^n - ax^n]$, the same subtractive number remains; if we subtract the subtractive number from an empty power $[0x^n - (-ax^n)]$, the same additive number remains. If we subtract an additive number from a subtractive number, the remainder is their subtractive sum; if we subtract a subtractive number from a greater subtractive number, the result is their subtractive difference; if the number from which one subtracts is smaller than the number subtracted, the result is their additive difference.[23]

Given these rules, al-Samaw'al could easily add and subtract polynomials by combining like terms. To multiply, of course, he needed the law of exponents. Al-Karajī had in essence used this law, as had Abū Kāmil and others. However, since the product of, for example, a square and a cube was expressed in words as a square-cube, the numerical property of adding exponents could not be seen. Al-Samaw'al decided that this law could best be expressed by using a table consisting of columns, each column representing a different power of either a number or an unknown. In fact, he also saw that he could deal with powers of $\frac{1}{x}$ as easily as with powers of x. In his work, the columns are headed by the Arabic letters standing for the numerals, reading both ways from the central column labeled 0. We will simply use the Arabic numerals themselves. Each column then has the name of the particular power or reciprocal power. For example, the column headed by a 2 on the left is named "square," that headed by a 5 on the left is named "square-cube," that headed by a 3 on the right is named "part of cube," and so on. To simplify matters we will just use powers of x. In his initial explanation of the rules, al-Samaw'al also put a particular number under the 1 on the left, such as 2, and then the various powers of 2 in the corresponding columns:

7	6	5	4	3	2	1	0	1	2	3	4	5	6	7
x^7	x^6	x^5	x^4	x^3	x^2	x	1	x^{-1}	x^{-2}	x^{-3}	x^{-4}	x^{-5}	x^{-6}	x^{-7}
128	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$

Al-Samaw'al now used the chart to explain what we call the law of exponents, $x^n x^m = x^{m+n}$: "The distance of the order of the product of the two factors from the order of one of the two factors is equal to the distance of the order of the other factor from the unit. If the factors are in different directions then we count (the distance) from the order of the first factor towards the unit; but, if they are in the same direction, we count away from the unit."[24] So, for example, to multiply x^3 by x^4, count four orders to the left of column 3 and get the result as x^7. To multiply x^3 by x^{-2}, count two orders to the right from column 3 and get the answer x^1. Using these rules, al-Samaw'al could easily multiply polynomials in x and $\frac{1}{x}$ as well as divide such polynomials by monomials.

Al-Samaw'al was also able to divide polynomials by polynomials using a similar chart. In this chart, which reminds us of the Chinese counting board as used in solving polynomial equations, each column again stands for a given power of x or of $\frac{1}{x}$. But now the numbers in each column represent the coefficients of the various polynomials involved in the division process. For example, to divide $20x^2 + 30x$ by $6x^2 + 12$, he first set the 20 and the 30 in the columns headed by x^2 and x, respectively, and the 6 and 12 below these in the columns headed respectively by x^2 and 1. Since there is an "empty order" for the divisor in the x column, he placed a 0 there. He next divided $20x^2$ by $6x^2$, getting 3 1/3, putting that number in the units column on the answer line. The product of 3 1/3 by $6x^2 + 12$ is $20x^2 + 40$. The next step is subtraction. The remainder in the x^2 column is naturally 0. In the x column the remainder is 30, while in the units column the remainder is -40. Al-Samaw'al now presented a new chart in which the 6, 0, 12, are shifted one place to the right, and the directions are given to divide that into $30x - 40$. The initial quotient of $30x$ by $6x^2$ is $5 \cdot 1/x$, so a 5 is placed in the answer line in the column headed by $\frac{1}{x}$, and the process is continued. We display here al-Samaw'al's first two charts for this division problem.

x^2	x	1	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{1}{x^3}$
		$3\frac{1}{3}$			
20	30				
6	0	12			

x^2	x	1	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{1}{x^3}$
		$3\frac{1}{3}$	5		
	30	-40			
	6	0	12		

In this particular example, the division was not exact. Al-Samaw'al continued the process through eight steps to get

$$3\frac{1}{3}+5\left(\frac{1}{x}\right)-6\frac{2}{3}\left(\frac{1}{x^2}\right)-10\left(\frac{1}{x^3}\right)+13\frac{1}{3}\left(\frac{1}{x^4}\right)+20\left(\frac{1}{x^5}\right)-26\frac{2}{3}\left(\frac{1}{x^6}\right)-40\left(\frac{1}{x^7}\right).$$

To show his fluency with the multiplication procedure, he then checked the answer by multiplying it by the divisor. Because the product differed from the dividend by terms only in $\frac{1}{x^6}$ and $\frac{1}{x^7}$, he called the result given "the answer approximately." Nevertheless, he also noted that there is a pattern to the coefficients of the quotient. In fact, if a_n represents the coefficient of $\frac{1}{x^n}$, the pattern is given by $a_{n+2} = -2a_n$. He then proudly wrote out the next 21 terms of the quotient, ending with $54{,}613\frac{1}{3}(\frac{1}{x^{28}})$.

Given that al-Samaw'al thought of extending division of polynomials into polynomials in $\frac{1}{x}$, and thought of partial results as approximations, it is not surprising that he would divide whole numbers by simply replacing x by 10. As already noted, al-Samaw'al was the first to explicitly recognize that one could approximate fractions more and more closely by calculating more and more decimal places. The work of al-Karajī and al-Samaw'al was thus extremely important in developing the idea that algebraic manipulations and manipulations with numbers are parallel. Virtually any technique that applies to one can be adapted to apply to the other.

9.3.4 Induction, Sums of Powers, and the Pascal Triangle

Another important idea introduced by al-Karajī and continued by al-Samaw'al and others was that of an inductive argument for dealing with certain arithmetic sequences. Thus, al-Karajī used such an argument to prove the result on the sums of integral cubes already known to Āryabhaṭa (and even, perhaps, to the Greeks). Al-Karajī did not, however, state a general result for arbitrary n. He stated his theorem for the particular integer 10:

$$1^3 + 2^3 + 3^3 + \cdots + 10^3 = (1 + 2 + 3 + \cdots + 10)^2.$$

His proof, nevertheless, was clearly designed to be extendable to any other integer.

Consider the square $ABCD$ with side $1 + 2 + 3 + \cdots + 10$ (Fig. 9.9). Setting $BB' = DD' = 10$, and completing the gnomon $BCDD'C'B'$, al-Karajī calculated the area of the

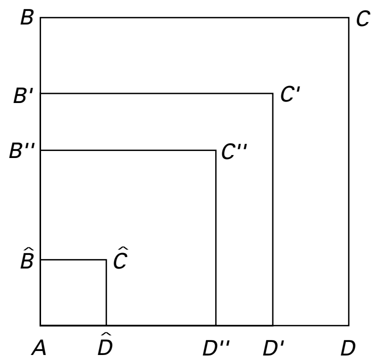

FIGURE 9.9
Al-Karajī's proof of the formula for the sum of the integral cubes

gnomon to be

$$2 \cdot 10(1 + 2 + \cdots + 9) + 10^2 = 2 \cdot 10 \cdot \frac{9 \cdot 10}{2} + 10^2 = 9 \cdot 10^2 + 10^2 = 10^3.$$

Since the area of square $ABCD$ is the sum of the areas of square $AB'C'D'$ and the gnomon, it follows that $(1 + 2 + \cdots + 10)^2 = (1 + 2 + \cdots + 9)^2 + 10^3$. A similar argument then shows that $(1 + 2 + \cdots + 9)^2 = (1 + 2 + \cdots + 8)^2 + 9^3$. Continuing in this way to the final square $A\hat{B}\hat{C}\hat{D}$ of area $1 = 1^3$, al-Karajī proved his theorem from the equality of square $ABCD$ to square $A\hat{B}\hat{C}\hat{D}$ plus the sum of the gnomons of areas $2^3,\ 3^3, \ldots,\ 10^3$.

Al-Karajī's argument included the two basic components of a modern argument by induction, namely, the truth of the statement for $n = 1$ $(1 = 1^3)$ and the deriving of the truth for $n = k$ from that for $n = k - 1$. Of course, this second component is not explicit since, in some sense, al-Karajī's argument is in reverse. That is, he starts from $n = 10$ and goes down to 1 rather than proceeding upward. Nevertheless, his argument in *al-Fakhrī* is the earliest extant proof of the sum formula for integral cubes.

The formulas for the sums of the integers and their squares had long been known, while the formula for the sum of cubes is easy to discover if one considers a few examples. To give an argument for their validity that generalizes to enable one to find a formula for the sum of fourth powers, however, is more difficult. Nonetheless, this was accomplished early in the eleventh century in a work by the Egyptian mathematician Abū 'Alī al-Ḥasan ibn al-Ḥasan ibn al-Haytham (965–1039). That he did not generalize his result to find the sums of higher powers is probably due to his needing only the formulas for the second and fourth powers in his computation of the volume of a paraboloid, to be discussed in Section 9.5.5.[25]

The central idea in ibn al-Haytham's proof of the sum formulas was the derivation of the equation

$$(n + 1)\sum_{i=1}^{n} i^k = \sum_{i=1}^{n} i^{k+1} + \sum_{p=1}^{n}\left(\sum_{i=1}^{p} i^k\right). \tag{9.2}$$

Ibn al-Haytham did not state this result in general form but only for particular integers, namely, $n = 4$ and $k = 1,\ 2,\ 3$. His proof, however, which, like al-Karajī's, used inductive