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Applied Partial Differential Equations with Fourier Series and Boundary Value Problems

5th Edition

Richard Haberman

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If we directly apply (5.30), then

$$(\lambda_2 - \lambda_1)\boldsymbol{u} \cdot \boldsymbol{v} = 0.$$

Thus, if $\lambda_1 \neq \lambda_2$ (different eigenvalues), the corresponding eigenvectors are orthogonal in the sense that

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0. \tag{5.31}$$

We leave as an exercise the proof that the eigenvalues of a symmetric real matrix are real.

EXAMPLE

The eigenvalues of the real symmetric matrix $\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$ are determined from $(6 - \lambda)(3 - \lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2) = 0$. For $\lambda = 2$, the eigenvector satisfies

$$6x_1 + 2x_2 = 2x_1 \quad \text{and} \quad 2x_1 + 3x_2 = 2x_2,$$

and hence $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. For $\lambda = 7$, it follows that

$$6x_1 + 2x_2 = 7x_1 \quad \text{and} \quad 2x_1 + 3x_2 = 7x_2,$$

and the eigenvector is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. As we have just proved for any real symmetric matrix, the eigenvectors are orthogonal, $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 - 2 = 0$.

Eigenvector expansions. *For real symmetric matrices* it can be shown that if an eigenvalue repeats R times, there will be R independent eigenvectors corresponding to that eigenvalue. These eigenvectors are automatically orthogonal to any eigenvectors corresponding to a different eigenvalue. The Gram–Schmidt procedure can be applied so that all R eigenvectors corresponding to the same eigenvalue can be constructed to be mutually orthogonal. In this manner, for real symmetric $n \times n$ matrices, n orthogonal eigenvectors can always be obtained. Since these vectors are orthogonal, they span the n-dimensional vector space and may be chosen as basis vectors. Any vector $\boldsymbol{v}$ may be represented in a series of the eigenvectors:

$$\boldsymbol{v} = \sum_{i=1}^{n} c_i \boldsymbol{\phi_i}, \tag{5.32}$$

where $\boldsymbol{\phi_i}$ is the ith eigenvector. For regular Sturm–Liouville eigenvalue problems, the eigenfunctions are complete, meaning that any (piecewise smooth) function can be represented in terms of an eigenfunction expansion

$$f(x) \sim \sum_{i=1}^{\infty} c_i \phi_i(x). \tag{5.33}$$

This is analogous to (5.32). In (5.33) the Fourier coefficients c_i are determined by the orthogonality of the eigenfunctions. Similarly, the coordinates c_i in (5.32) are determined by the orthogonality of the eigenvectors. We dot Equation (5.32) into $\boldsymbol{\phi_m}$:

$$\boldsymbol{v} \cdot \boldsymbol{\phi_m} = \sum_{i=1}^{n} c_i \boldsymbol{\phi_i} \cdot \boldsymbol{\phi_m} = c_m \boldsymbol{\phi_m} \cdot \boldsymbol{\phi_m},$$

since $\boldsymbol{\phi_i} \cdot \boldsymbol{\phi_m} = 0, i \neq m$, determining c_m.

Linear systems. Sturm–Liouville eigenvalue problems arise in separating variables for partial differential equations. One way in which the matrix eigenvalue problem occurs is in "separating" a linear homogeneous system of ordinary differential equations with constant coefficients. We will be *very* brief. A linear homogeneous first-order system of differential equations may be represented by

$$\boxed{\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}\boldsymbol{v},} \tag{5.34}$$

where $\boldsymbol{A}$ is an $n \times n$ matrix and $\boldsymbol{v}$ is the desired n-dimensional vector solution. $\boldsymbol{v}$ usually satisfies given initial conditions, $\boldsymbol{v}(0) = \boldsymbol{v}_0$. We seek special solutions of the form of simple exponentials:

$$\boxed{\boldsymbol{v}(t) = e^{\lambda t}\boldsymbol{\phi},} \tag{5.35}$$

where $\boldsymbol{\phi}$ is a constant vector. This is analogous to seeking product solutions by the method of separation of variables. Since $d\boldsymbol{v}/dt = \lambda e^{\lambda t}\boldsymbol{\phi}$, it follows that

$$\boldsymbol{A}\boldsymbol{\phi} = \lambda\boldsymbol{\phi}. \tag{5.36}$$

Thus, there exist solutions to (5.34) of the form (5.35) if λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{\phi}$ is a corresponding eigenvector. We now restrict our attention to *real symmetric matrices* $\boldsymbol{A}$. There will always be n mutually orthogonal eigenvectors $\boldsymbol{\phi}_i$. We have obtained n special solutions to the linear homogeneous system (5.34). A *principle of superposition* exists, and hence a linear combination of these solutions also satisfies (5.34):

$$\boxed{\boldsymbol{v} = \sum_{i=1}^{n} c_i e^{\lambda_i t}\boldsymbol{\phi}_i.} \tag{5.37}$$

We attempt to determine c_i so that (5.37) satisfies the initial conditions, $\boldsymbol{v}(0) = \boldsymbol{v}_0$:

$$\boldsymbol{v}_0 = \sum_{i=1}^{n} c_i \boldsymbol{\phi_i}.$$

Here, the orthogonality of the eigenvectors is helpful, and thus, as before,

$$c_i = \frac{\boldsymbol{v}_0 \cdot \boldsymbol{\phi_i}}{\boldsymbol{\phi_i} \cdot \boldsymbol{\phi_i}}.$$

EXERCISES 5 APPENDIX

5A.1. Prove that the eigenvalues of real symmetric matrices are real.

5A.2. **(a)** Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

has only one independent eigenvector.

(b) Show that the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has two independent eigenvectors.

5A.3. Consider the eigenvectors of the matrix

$$\boldsymbol{A} = \begin{bmatrix} 6 & 4 \\ 1 & 3 \end{bmatrix}.$$

(a) Show that the eigenvectors are not orthogonal.

(b) If the "dot product" of two vectors is defined as follows,

$$\boldsymbol{a} \cdot \boldsymbol{b} = \frac{1}{4} a_1 b_1 + a_2 b_2,$$

show that the eigenvectors are orthogonal with this dot product.

5A.4. Solve $d\boldsymbol{v}/dt = \boldsymbol{A}\boldsymbol{v}$ using matrix methods if

*__(a)__ $\boldsymbol{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}, \quad \boldsymbol{v}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \boldsymbol{v}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

5A.5. Show that the eigenvalues are real and the eigenvectors orthogonal:

(a) $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}$

*__(b)__ $\boldsymbol{A} = \begin{bmatrix} 3 & 1-i \\ 1+i & 1 \end{bmatrix}$ (see Exercise 5A.6)

5A.6. For a matrix $\boldsymbol{A}$ whose entries are complex numbers, the complex conjugate of the transpose is denoted by $\boldsymbol{A}^H$. For matrices in which $\boldsymbol{A}^H = \boldsymbol{A}$ (called **Hermitian**):

(a) Prove that the eigenvalues are real.

(b) Prove that eigenvectors corresponding to different eigenvalues are orthogonal (in the sense that $\boldsymbol{\phi_i} \cdot \overline{\boldsymbol{\phi_m}} = 0$, where $\overline{}$ denotes the complex conjugate).

6 RAYLEIGH QUOTIENT

The Rayleigh quotient can be derived from the Sturm–Liouville differential equation,

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0, \tag{6.1}$$

by multiplying (6.1) by ϕ and integrating:

$$\int_a^b \left[\phi\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi^2\right]\ dx + \lambda\int_a^b \phi^2\sigma\ dx = 0.$$

Since $\int_a^b \phi^2\sigma\ dx > 0$, we can solve for λ:

$$\lambda = \frac{-\int_a^b \left[\phi\frac{d}{dx}\left(p\frac{d\phi}{dx}\right) + q\phi^2\right]\ dx}{\int_a^b \phi^2\sigma\ dx}. \tag{6.2}$$

Integration by parts [$\int u\,dv = uv - \int v\,du$, where $u = \phi,\ dv = d/dx(p\,d\phi/dx)\,dx$ and hence $du = d\phi/dx\ dx,\ v = p\,d\phi/dx$] yields an expression involving the function ϕ evaluated at the boundary:

$$\boxed{\lambda = \frac{-p\phi\frac{d\phi}{dx}\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\phi^2\right]\ dx}{\int_a^b \phi^2\sigma\ dx},} \tag{6.3}$$

known as the **Rayleigh quotient**. In Sections 3 and 4 we have indicated some applications of this result. Further discussion will be given in Section 7.

Nonnegative eigenvalues. Often in physical problems, the sign of λ is quite important. As shown in Section 2.1, $dh/dt + \lambda h = 0$ in certain heat flow problems. Thus, positive λ corresponds to exponential decay in time, while negative λ corresponds to exponential growth. On the other hand, in certain vibration problems (see Section 7), $d^2h/dt^2 = -\lambda h$. There, only positive λ corresponds to the "usually" expected oscillations. Thus, in both types of problems we often expect $\lambda \geq 0$:

> The Rayleigh quotient (6.3) directly proves that $\lambda \geq 0$ if
> (a) $-p\,\phi\frac{d\phi}{dx}\Big|_a^b \geq 0$, and
> (b) $q \leq 0$. (6.4)

We claim that both (a) and (b) are physically reasonable conditions for nonnegative λ. Consider the boundary constraint, $-p\phi\ d\phi/dx|_a^b \geq 0$. The simplest types of homogeneous boundary conditions, $\phi = 0$ and $d\phi/dx = 0$, do not contribute to this boundary term, satisfying (a). The condition $d\phi/dx = h\phi$ (for the physical cases of Newton's law of cooling or the elastic boundary condition) has $h > 0$ at the left end, $x = a$. Thus, it will have a positive contribution at $x = a$. The sign switch at the right end, which occurs for this type of boundary condition, will also cause a positive contribution. The periodic boundary condition [e.g., $\phi(a) = \phi(b)$ and $p(a)\ d\phi/dx(a) = p(b)\ d\phi/dx(b)$] as well as the singularity condition [$\phi(a)$ bounded, if $p(a) = 0$] also do not contribute. Thus, in all these cases $-p\phi\ d\phi/dx|_a^b \geq 0$.

The source constraint $q \leq 0$ also has a meaning in physical problems. For heat flow problems, $q \leq 0$ corresponds ($q = \alpha,\ Q = \alpha u$) to an energy-absorbing (endothermic) reaction, while for vibration problems, $q \leq 0$ corresponds ($q = \alpha,\ Q = \alpha u$) to a restoring force.

Minimization principle. The Rayleigh quotient cannot be used to determine explicitly the eigenvalue (since ϕ is unknown). Nonetheless, it can be quite useful in estimating the eigenvalues. This is because of the following theorem: **The minimum value of the Rayleigh quotient for all continuous functions satisfying the boundary conditions** (but not necessarily the differential equation) **is the lowest eigenvalue:**

$$\boxed{\lambda_1 = \min \frac{-pu\ du/dx|_a^b + \displaystyle\int_a^b \left[p\left(du/dx\right)^2 - qu^2\right]\ dx}{\displaystyle\int_a^b u^2\sigma\ dx},} \tag{6.5}$$

where λ_1 represents the smallest eigenvalue. The minimization includes all continuous functions that satisfy the boundary conditions. The minimum is obtained only for $u = \phi_1(x)$, the lowest eigenfunction. For example, the lowest eigenvalue is important in heat flow problems (see Section 4).

Trial functions. Before proving (6.5), we will indicate how (6.5) is applied to obtain bounds on the lowest eigenvalue. Equation (6.5) is difficult to apply directly since we do not know how to minimize over all functions. However, let u_T be *any* continuous function satisfying the boundary conditions; u_T is known as a **trial function.** We compute the Rayleigh quotient of this trial function, $RQ[u_T]$:

$$\lambda_1 \leq RQ[u_T] = \frac{-pu_T\ du_T/dx|_a^b + \int_a^b \left[p\left(du_T/dx\right)^2 - qu_T^2\right]\ dx}{\int_a^b u_T^2\sigma\ dx}. \tag{6.6}$$

We have noted that λ_1 must be less than or equal to the quotient since λ_1 is the minimum of the ratio for all functions. Equation (6.6) gives an **upper bound** for the lowest eigenvalue.

EXAMPLE

Consider the well-known eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$
$$\phi(0) = 0$$
$$\phi(1) = 0.$$

We already know that $\lambda = n^2\pi^2 (L = 1)$, and hence the lowest eigenvalue is $\lambda_1 = \pi^2$. For this problem, the Rayleigh quotient simplifies, and (6.6) becomes

$$\lambda_1 \leq \frac{\int_0^1 (du_T/dx)^2\ dx}{\int_0^1 u_T^2\ dx}. \tag{6.7}$$

Trial functions must be continuous and satisfy the homogeneous boundary conditions, in this case, $u_T(0) = 0$ and $u_T(1) = 0$. In addition, we claim that the closer the trial function is to the actual eigenfunction, the more accurate is the bound of the lowest eigenvalue. Thus, we also choose trial functions with no zeros in the interior, since we already know theoretically that the lowest eigenfunction does not have a zero. We will compute the Rayleigh quotient for the three trial functions sketched in Fig. 6.1. For

$$u_T = \begin{cases} x, & x < \frac{1}{2} \\ 1 - x, & x > \frac{1}{2}, \end{cases}$$

(6.7) becomes

$$\lambda_1 \leq \frac{\int_0^{1/2}\ dx + \int_{1/2}^1\ dx}{\int_0^{1/2} x^2\ dx + \int_{1/2}^1 (1-x)^2\ dx} = \frac{1}{\frac{1}{24} + \frac{1}{24}} = 12,$$

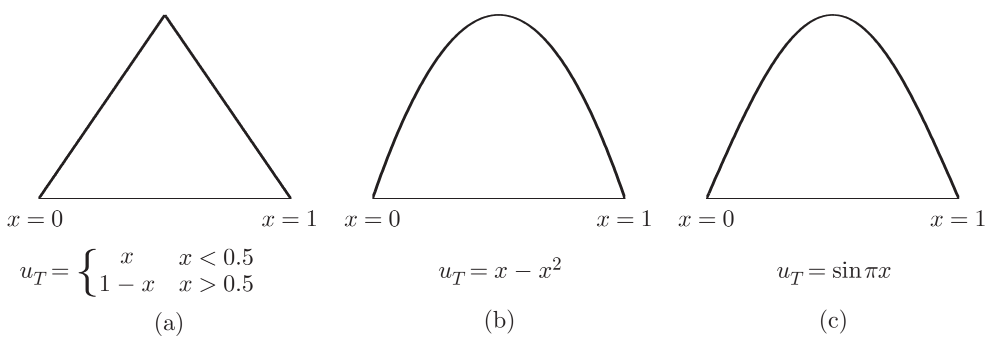

FIGURE 6.1 Trial functions: are continuous, satisfy the boundary conditions, and are of one sign.