PEARSON NEW INTERNATIONAL EDITION

Introduction to Robotics: Mechanics and Control

John J. Craig

Third Edition

ALWAYS LEARNING

PEARSON

Pearson New International Edition

Introduction to Robotics:
Mechanics and Control
John J. Craig
Third Edition

PEARSON®

Section 4.7 gives an example of *projecting* a general goal into the subspace of a manipulator with five degrees of freedom in order to compute joint angles that will result in the manipulator's reaching the attainable frame nearest to the desired one.

4.4 ALGEBRAIC VS. GEOMETRIC

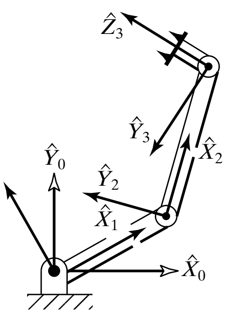
As an introduction to solving kinematic equations, we will consider two different approaches to the solution of a simple planar three-link manipulator.

Algebraic solution

Consider the three-link planar manipulator introduced in Chapter 3. It is shown with its link parameters in Fig. 4.7.

Following the method of Chapter 3, we can use the link parameters easily to find the kinematic equations of this arm:

$$ {}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (4.6) $$



i	$\alpha_i - 1$	$\alpha_i - 1$	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

FIGURE 4.7: Three-link planar manipulator and its link parameters.

To focus our discussion on inverse kinematics, we will assume that the necessary transformations have been performed so that the goal point is a specification of the wrist frame relative to the base frame, that is, ${}^B_W T$. Because we are working with a planar manipulator, specification of these goal points can be accomplished most easily by specifying three numbers: x, y, and ϕ, where ϕ is the orientation of link 3 in the plane (relative to the $+\hat{X}$ axis). Hence, rather than giving a general ${}^B_W T$ as a goal specification, we will assume a transformation with the structure

$$ {}^B_W T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.7} $$

All attainable goals must lie in the subspace implied by the structure of equation (4.7). By equating (4.6) and (4.7), we arrive at a set of four nonlinear equations that must be solved for θ_1, θ_2, and θ_3:

$$ c_\phi = c_{123}, \tag{4.8} $$

$$ s_\phi = s_{123}, \tag{4.9} $$

$$ x = l_1 c_1 + l_2 c_{12}, \tag{4.10} $$

$$ y = l_1 s_1 + l_2 s_{12}. \tag{4.11} $$

We now begin our algebraic solution of equations (4.8) through (4.11). If we square both (4.10) and (4.11) and add them, we obtain

$$ x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2, \tag{4.12} $$

where we have made use of

$$ \begin{aligned} c_{12} &= c_1 c_2 - s_1 s_2, \\ s_{12} &= c_1 s_2 + s_1 c_2. \end{aligned} \tag{4.13} $$

Solving (4.12) for c_2, we obtain

$$ c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}. \tag{4.14} $$

In order for a solution to exist, the right-hand side of (4.14) must have a value between -1 and 1. In the solution algorithm, this constraint would be checked at this time to find out whether a solution exists. Physically, if this constraint is not satisfied, then the goal point is too far away for the manipulator to reach.

Assuming the goal is in the workspace, we write an expression for s_2 as

$$ s_2 = \pm\sqrt{1 - c_2^2}. \tag{4.15} $$

Finally, we compute θ_2, using the two-argument arctangent routine[1]:

$$ \theta_2 = \text{Atan2}(s_2, c_2). \tag{4.16} $$

[1] See Section 2.8.

The choice of signs in (4.15) corresponds to the multiple solution in which we can choose the "elbow-up" or the "elbow-down" solution. In determining θ_2, we have used one of the recurring methods for solving the type of kinematic relationships that often arise, namely, to determine both the sine and cosine of the desired joint angle and then apply the two-argument arctangent. This ensures that we have found all solutions and that the solved angle is in the proper quadrant.

Having found θ_2, we can solve (4.10) and (4.11) for θ_1. We write (4.10) and (4.11) in the form

$$x = k_1 c_1 - k_2 s_1, \tag{4.17}$$

$$y = k_1 s_1 + k_2 c_1, \tag{4.18}$$

where

$$\begin{aligned} k_1 &= l_1 + l_2 c_2, \\ k_2 &= l_2 s_2. \end{aligned} \tag{4.19}$$

In order to solve an equation of this form, we perform a change of variables. Actually, we are changing the way in which we write the constants k_1 and k_2.

If

$$r = +\sqrt{k_1^2 + k_2^2} \tag{4.20}$$

and

$$\gamma = \text{Atan2}(k_2, k_1),$$

then

$$\begin{aligned} k_1 &= r \cos\gamma, \\ k_2 &= r \sin\gamma. \end{aligned} \tag{4.21}$$

Equations (4.17) and (4.18) can now be written as

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1, \tag{4.22}$$

$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1, \tag{4.23}$$

so

$$\cos(\gamma + \theta_1) = \frac{x}{r}, \tag{4.24}$$

$$\sin(\gamma + \theta_1) = \frac{y}{r}. \tag{4.25}$$

Using the two-argument arctangent, we get

$$\gamma + \theta_1 = \text{Atan2}\left(\frac{y}{r}, \frac{x}{r}\right) = \text{Atan2}(y, x), \tag{4.26}$$

and so

$$\theta_1 = \text{Atan2}(y, x) - \text{Atan2}(k_2, k_1). \tag{4.27}$$

Note that, when a choice of sign is made in the solution of θ_2 above, it will cause a sign change in k_2, thus affecting θ_1. The substitutions used, (4.20) and (4.21), constitute a method of solution of a form appearing frequently in kinematics—namely, that of (4.10) or (4.11). Note also that, if $x = y = 0$, then (4.27) becomes undefined—in this case, θ_1 is arbitrary.

Finally, from (4.8) and (4.9), we can solve for the sum of θ_1 through θ_3:

$$\theta_1 + \theta_2 + \theta_3 = \text{Atan2}(s_\phi, c_\phi) = \phi. \tag{4.28}$$

From this, we can solve for θ_3, because we know the first two angles. It is typical with manipulators that have two or more links moving in a plane that, in the course of solution, expressions for sums of joint angles arise.

In summary, an algebraic approach to solving kinematic equations is basically one of manipulating the given equations into a form for which a solution is known. It turns out that, for many common geometries, several forms of transcendental equations commonly arise. We have encountered a couple of them in this preceding section. In Appendix C, more are listed.

Geometric solution

In a geometric approach to finding a manipulator's solution, we try to decompose the spatial geometry of the arm into several plane-geometry problems. For many manipulators (particularly when the $\alpha_i = 0$ or ± 90) this can be done quite easily. Joint angles can then be solved for by using the tools of plane geometry [7]. For the arm with three degrees of freedom shown in Fig. 4.7, because the arm is planar, we can apply plane geometry directly to find a solution.

Figure 4.8 shows the triangle formed by l_1, l_2, and the line joining the origin of frame {0} with the origin of frame {3}. The dashed lines represent the other possible configuration of the triangle, which would lead to the same position of the frame {3}. Considering the solid triangle, we can apply the "law of cosines" to solve for θ_2:

$$x^2 + y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos(180 + \theta_2). \tag{4.29}$$

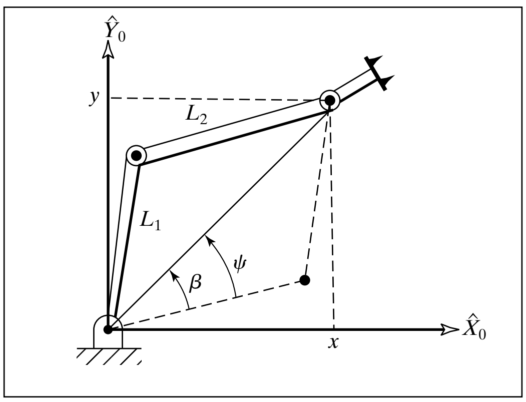


FIGURE 4.8: Plane geometry associated with a three-link planar robot.

Now; $\cos(180 + \theta_2) = -\cos(\theta_2)$, so we have

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}. \tag{4.30}$$

In order for this triangle to exist, the distance to the goal point $\sqrt{x^2 + y^2}$ must be less than or equal to the sum of the link lengths, $l_1 + l_2$. This condition would be checked at this point in a computational algorithm to verify existence of solutions. This condition is not satisfied when the goal point is out of reach of the manipulator. Assuming a solution exists, this equation is solved for that value of θ_2 that lies between 0 and −180 degrees, because only for these values does the triangle in Fig. 4.8 exist. The other possible solution (the one indicated by the dashed-line triangle) is found by symmetry to be $\theta_2' = -\theta_2$.

To solve for θ_1, we find expressions for angles ψ and β as indicated in Fig. 4.8. First, β may be in any quadrant, depending on the signs of x and y. So we must use a two-argument arctangent:

$$\beta = \text{Atan2}(y, x). \tag{4.31}$$

We again apply the law of cosines to find ψ:

$$\cos\psi = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2 l_1 \sqrt{x^2 + y^2}}. \tag{4.32}$$

Here, the arccosine must be solved so that $0 \le \psi \le 180°$, in order that the geometry which leads to (4.32) will be preserved. These considerations are typical when using a geometric approach—we must apply the formulas we derive only over a range of variables such that the geometry is preserved. Then we have

$$\theta_1 = \beta \pm \psi, \tag{4.33}$$

where the plus sign is used if $\theta_2 < 0$ and the minus sign if $\theta_2 > 0$.

We know that angles in a plane add, so the sum of the three joint angles must be the orientation of the last link:

$$\theta_1 + \theta_2 + \theta_3 = \phi. \tag{4.34}$$

This equation is solved for θ_3 to complete our solution.

4.5 ALGEBRAIC SOLUTION BY REDUCTION TO POLYNOMIAL

Transcendental equations are often difficult to solve because, even when there is only one variable (say, θ), it generally appears as $\sin\theta$ and $\cos\theta$. Making the following substitutions, however, yields an expression in terms of a single variable, u:

$$\begin{aligned} u &= \tan\frac{\theta}{2}, \\ \cos\theta &= \frac{1 - u^2}{1 + u^2}, \\ \sin\theta &= \frac{2u}{1 + u^2}. \end{aligned} \tag{4.35}$$

This is a very important geometric substitution used often in solving kinematic equations. These substitutions convert transcendental equations into polynomial equations in u. Appendix A lists these and other trigonometric identities.

EXAMPLE 4.3

Convert the transcendental equation

$$a \cos\theta + b \sin\theta = c \tag{4.36}$$

into a polynomial in the tangent of the half angle, and solve for θ.

Substituting from (4.35) and multiplying through by $1 + u^2$, we have

$$a(1 - u^2) + 2bu = c(1 + u^2). \tag{4.37}$$

Collecting powers of u yields

$$(a + c)u^2 - 2bu + (c - a) = 0, \tag{4.38}$$

which is solved by the quadratic formula:

$$u = \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a + c}. \tag{4.39}$$

Hence,

$$\theta = 2 \tan^{-1}\left(\frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a + c}\right). \tag{4.40}$$

Should the solution for u from (4.39) be complex, there is no real solution to the original transcendental equation. Note that, if $a + c = 0$, the argument of the arctangent becomes infinity and hence $\theta = 180°$. In a computer implementation, this potential division by zero should be checked for ahead of time. This situation results when the quadratic term of (4.38) vanishes, so that the quadratic degenerates into a linear equation.

Polynomials up to degree four possess closed-form solutions [8, 9], so manipulators sufficiently simple that they can be solved by algebraic equations of this degree (or lower) are called **closed-form-solvable** manipulators.

4.6 PIEPER'S SOLUTION WHEN THREE AXES INTERSECT

As mentioned earlier, although a completely general robot with six degrees of freedom does not have a closed-form solution, certain important special cases can be solved. Pieper [3, 4] studied manipulators with six degrees of freedom in which three consecutive axes intersect at a point.[2] In this section, we outline the method he developed for the case of all six joints revolute, with the last three axes intersecting. His method applies to other configurations, which include prismatic

[2]Included in this family of manipulators are those with three consecutive parallel axes, because they meet at the point at infinity.