Pearson New International Edition

# Fundamentals of Engineering Electromagnetics

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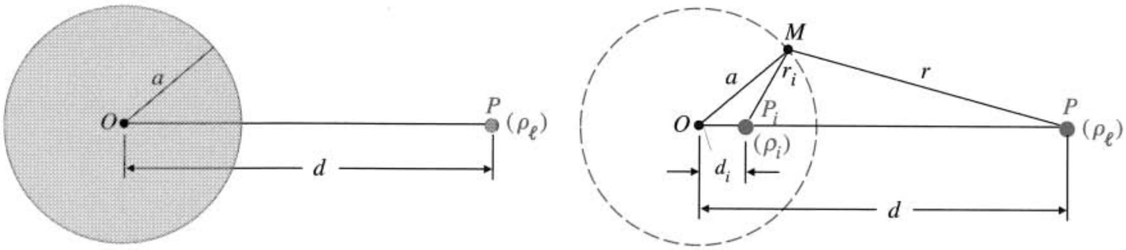


(a) Line charge and parallel conducting cylinder.

(b) Line charge and its image.

**FIGURE 3-28** Cross section of line charge and its image in a parallel, conducting, circular cylinder.

$$V = -\int_{r_0}^{r} E_r\, dr = -\frac{\rho_\ell}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r}\,dr = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r}. \tag{3-157}$$

Note that the reference point for zero potential, $r_0$, cannot be at infinity because setting $r_0 = \infty$ in Eq. (3-157) would make $V$ infinite everywhere else. Let us leave $r_0$ unspecified for the time being. The potential at a point on or outside the cylindrical surface is obtained by adding the contributions of $\rho_\ell$ and $\rho_i$. In particular, at a point $M$ on the cylindrical surface shown in Fig. 3-28(b) we have

$$V_M = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r} - \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_i}{r}. \tag{3-158}$$

In Eq. (3-158) we have chosen, for simplicity, a point equidistant from $\rho_\ell$ and $\rho_i$ as the reference point for zero potential so that the $\ln r_0$ terms cancel. Otherwise, a constant term should be included in the right side of Eq. (3-158), but it would not affect what follows. Equipotential surfaces are specified by

$$\frac{r_i}{r} = \text{Constant}. \tag{3-159}$$

If an equipotential surface is to coincide with the cylindrical surface ($\overline{OM} = a$), the point $P_i$ must be located in such a way as to make triangles $OMP_i$ and $OPM$ similar. These two triangles already have one common angle, $\angle MOP_i$. Point $P_i$ should be chosen to make $\angle OMP_i = \angle OPM$. We have

$$\frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}},$$

or

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant}. \tag{3-160}$$

From Eq. (3-160) we see that if

$$\boxed{d_i = \frac{a^2}{d}} \tag{3-161}$$

the image line charge $-\rho_\ell$, together with $\rho_\ell$, will make the dashed cylindrical surface in Fig. 3-28(b) equipotential. As the point $M$ changes its location on the dashed circle, both $r_i$ and $r$ will change; but their ratio remains a constant that equals $a/d$. Point $P_i$ is called the ***inverse point*** of $P$ with respect to a circle of radius $a$.

The image line charge $\rho_i = -\rho_\ell$ can then replace the cylindrical conducting surface, and $V$ and $\mathbf{E}$ at any point *outside the surface* can be determined from the line charges $\rho_\ell$ and $-\rho_\ell$. By symmetry we find that the parallel cylindrical surface surrounding the original line charge $\rho_\ell$ with radius $a$ and its axis at a distance $d_i$ to the right of $P$ is also an equipotential surface. This observation enables us to calculate the capacitance per unit length of an open-wire transmission line consisting of two parallel conductors of circular cross section.

**EXAMPLE 3-25**

Determine the capacitance per unit length between two long, parallel, circular conducting wires of radius $a$. The axes of the wires are separated by a distance $D$.

**SOLUTION**

Refer to the cross section of the two-wire transmission line shown in Fig. 3-29. The equipotential surfaces of the two wires can be considered to have been generated by a pair of line charges $+\rho_\ell$ and $-\rho_\ell$ separated by a distance $(D - 2d_i) = d - d_i$. The potential difference between the two wires is that between any two points on the respective wires. Let subscripts 1 and 2 denote the wires surrounding the equivalent line charges $+\rho_\ell$ and $-\rho_\ell$, respectively. We have, from Eqs. (3-158) and (3-160),

$$V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{a}{d}$$

and, similarly,

$$V_1 = -\frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{a}{d}.$$

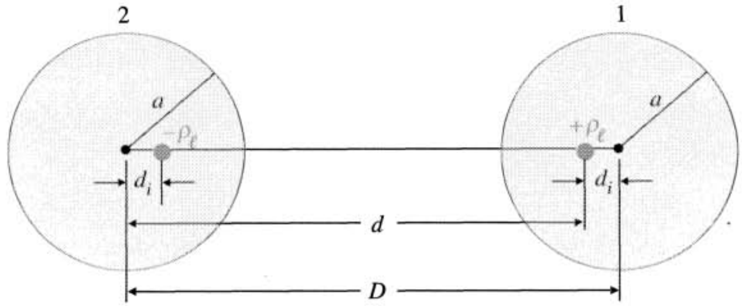


FIGURE 3-29 Cross section of two-wire transmission line and equivalent line charges (Example 3-25).

We note that $V_1$ is a positive quantity, whereas $V_2$ is negative because $a < d$. The capacitance per unit length is

$$C = \frac{\rho_\ell}{V_1 - V_2} = \frac{\pi\epsilon_0}{\ln(d/a)}, \tag{3-162}$$

where

$$d = D - d_i = D - \frac{a^2}{d},$$

from which we obtain[†]

$$d = \tfrac{1}{2}(D + \sqrt{D^2 - 4a^2}). \tag{3-163}$$

Using Eq. (3-163) in Eq. (3-162), we have

**Capacitance per unit length of parallel wires**

$$\boxed{C = \frac{\pi\epsilon_0}{\ln\left[(D/2a) + \sqrt{(D/2a)^2 - 1}\right]} \quad \text{(F/m)}.} \tag{3-164}$$

Since

$$\ln\left[x + \sqrt{x^2 - 1}\right] = \cosh^{-1}x \text{ for } x > 1,$$

Eq. (3-164) can be written alternatively as

$$\boxed{C = \frac{\pi\epsilon_0}{\cosh^{-1}(D/2a)} \quad \text{(F/m)}.} \tag{3-165}$$

[†] The other solution, $d = \frac{1}{2}(D - \sqrt{D^2 - 4a^2})$, is discarded because both $D$ and $d$ are usually much larger than $a$.

When the diameter of the wires is very small in comparison with the distance of separation, $(D/2a) \gg 1$, Eq. (3-164) simplifies to

$$C = \frac{\pi\epsilon_0}{\ln(D/a)} \qquad \text{(F/m)}. \tag{3-166}$$

**EXERCISE 3.22** A long power transmission line, 2 (cm) in radius, is parallel to and situated 10 (m) above the ground. Assuming the ground to be an infinite flat conducting plane, find the capacitance per meter of the line with respect to the ground.

**ANS.** 8.04 (pF/m).

## REVIEW QUESTIONS

Q.3-27 Write Poisson's and Laplace's equations in vector notation for a simple medium.

Q.3-28 Write Poisson's and Laplace's equations in Cartesian coordinates for a simple medium.

Q.3-29 If $\nabla^2 U = 0$, why does it not follow that $U$ is identically zero?

Q.3-30 A fixed voltage is connected across a parallel-plate capacitor.

**a)** Does the electric field intensity in the space between the plates depend on the permittivity of the medium?

**b)** Does the electric flux density depend on the permittivity of the medium? Explain.

Q.3-31 Assume that fixed charges $+Q$ and $-Q$ are deposited on the plates of an isolated parallel-plate capacitor.

**a)** Does the electric field intensity in the space between the plates depend on the permittivity of the medium?

**b)** Does the electric flux density depend on the permittivity of the medium? Explain.

Q.3-32 State in words the *uniqueness theorem of electrostatics.*

Q.3-33 What is the image of a spherical cloud of electrons with respect to an infinite conducting plane?

Q.3-34 What is the image of an infinitely long line charge of density $\rho_\ell$ with respect to a parallel conducting circular cylinder?

Q.3-35 Where is the zero-potential surface of the two-wire transmission line in Fig. 3-29?

## REMARKS

1. Poisson's Eq. (3-126) and Laplace's Eq. (3-130) do not hold if the medium is nonlinear, inhomogeneous, or anisotropic.
2. The method of images can be used to determine the fields *only* in the region where the image charges are *not* located.

## SUMMARY


This chapter deals with the static electric fields of charges that are at rest and do not change with time. After having defined the electric field intensity **E** as the force per unit charge, we

- presented the two fundamental postulates of electrostatics in free space that specify the divergence and the curl of **E**,
- derived Coulomb's law and Gauss's law, which enabled us to determine the electric field due to discrete and continuous charge distributions,
- introduced the concept of the scalar electric potential,
- considered the effect of material media on static electric field,
- discussed the macroscopic effect of induced dipoles by finding the equivalent polarization charge densities,
- defined electric flux density or electric displacement, **D**, and the dielectric constant,
- discussed the boundary conditions for static electric fields,
- defined capacitance and explained the procedure for its determination,
- found the formulas for stored electrostatic energy,
- used the principle of virtual displacement to calculate the force on an object in a charged system,
- introduced Poisson's and Laplace's equations and illustrated the method of solution for simple problems, and
- explained the method of images for solving electrostatic boundary-value problems.


## PROBLEMS

**P.3-1** The cathode-ray oscilloscope (CRO) shown in Fig. 3-2 is used to measure the voltage applied to the parallel deflection plates.

**a)** Assuming no breakdown in insulation, what is the maximum voltage that can be measured if the distance of separation between the plates is $h$?

**b)** What is the restriction on $L$ if the diameter of the screen is $D$?

**c)** What can be done with a fixed geometry to double the CRO's maximum measurable voltage?

**P.3-2** Three 2-($\mu$C) point charges are located in air at the corners of an equilateral triangle that is 10 (cm) on each side. Find the magnitude and direction of the force experienced by each charge.

**P.3-3** Two point charges, $Q_1$ and $Q_2$, are located at (0, 5, $-1$) and (0, $-2$, 6), respectively. Find the relation between $Q_1$ and $Q_2$ such that the total force on a test charge at the point $P(0, 2, 3)$ will have

**a)** no $y$-component, and

**b)** no $z$-component.

**P.3-4** Three point charges $Q_1 = -9$ ($\mu$C), $Q_2 = 4$ ($\mu$C), and $Q_3 = -36$ ($\mu$C) are arranged on a straight line. The distance between $Q_1$ and $Q_3$ is 9 (cm). It is claimed that a location can be selected for $Q_2$ such that each charge will experience a zero force. Find this location.

**P.3-5** In Example 3-8 determine the position of the point $P$ on the $z$-axis beyond which the disk may be regarded as a point charge if the error in the calculation of **E** is not more than 1%.

**P.3-6** A line charge of uniform charge density $\rho_\ell$ forms a circle of radius $b$ that lies in the $xy$-plane in air with its center at the origin.

**a)** Find the electric field intensity **E** at the point $(0, 0, h)$.

**b)** At what value of $h$ will **E** in part (a) be a maximum? What is this maximum?

**c)** Explain why **E** has a maximum at that location.

**P.3-7** A line charge of uniform density $\rho_\ell$ forms a semicircle of radius $b$ in the upper half $xy$-plane. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

**P.3-8** A spherical distribution of charge $\rho = \rho_0[1 - (R^2/b^2)]$ exists in the region $0 \leqslant R \leqslant b$. This charge distribution is concentrically surrounded by a conducting shell with inner radius $R_i (> b)$ and outer radius $R_o$. Determine **E** everywhere.

**P.3-9** Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b$ $(b > a)$, carry surface charge densities $\rho_{sa}$ and $\rho_{sb}$, respectively.

**a)** Determine **E** everywhere.

**b)** What must be the relation between $a$ and $b$ in order that **E** vanishes for $r > b$?

**P.3-10** Determine the work done in carrying a $+5$ ($\mu$C) charge from $P_1(1, 2, -4)$ to $P_2(-2, 8, -4)$ in the field $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$

**a)** along the parabola $y = 2x^2$, and

**b)** along the straight line joining $P_1$ and $P_2$.

**P.3-11** Repeat problem P.3-10 if the field is $\mathbf{E} = \mathbf{a}_x y - \mathbf{a}_y x$.

**P.3-12** A finite line charge of length $L$ carrying uniform line charge density $\rho_\ell$ is coincident with the $x$-axis.

**a)** Determine $V$ in the plane bisecting the line charge.

**b)** Determine **E** from $\rho_\ell$ directly by applying Coulomb's law.

**c)** Check the answer in part (b) with $-\nabla V$.