PEARSON NEW INTERNATIONAL EDITION

Field and Wave Electromagnetics

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Second Edition

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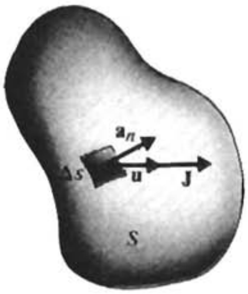

FIGURE 5–1
Conduction current due to drift motion of charge carriers across a surface.

Since current is the time rate of change of charge, we have

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s = Nq\mathbf{u} \cdot \Delta\mathbf{s} \qquad \text{(A)}. \tag{5–2}$$

In Eq. (5–2) we have written $\Delta\mathbf{s} = \mathbf{a}_n \Delta s$ as a vector quantity. It is convenient to define a vector point function, ***volume current density***, or simply ***current density***, $\mathbf{J}$, in amperes per *square* meter,

$$\mathbf{J} = Nq\mathbf{u} \qquad (\text{A/m}^2), \tag{5–3}$$

so that Eq. (5–2) can be written as

$$\Delta I = \mathbf{J} \cdot \Delta\mathbf{s}. \tag{5–4}$$

The total current I flowing through an arbitrary surface S is then the flux of the $\mathbf{J}$ vector through S:

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A).}} \tag{5–5}$$

Noting that the product Nq is in fact free charge per unit volume, we may rewrite Eq. (5–3) as

$$\boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\text{A/m}^2),} \tag{5–6}$$

which is the relation between the ***convection current density*** and the velocity of the charge carrier.

EXAMPLE 5–1 In vacuum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0.

Solution The region between the cathode and the anode is shown in Fig. 5–2, where a cloud of electrons (negative space charge) exists such that the force of repulsion makes the electrons boiled off the hot cathode leave essentially with a zero velocity. In other words, the net electric field at the cathode is zero. Neglecting fringing effects, we have

$$\mathbf{E}(0) = \mathbf{a}_y E_y(0) = -\mathbf{a}_y \left.\frac{dV(y)}{dy}\right|_{y=0} = 0. \tag{5–7}$$

In the steady state the current density is constant, independent of y:

$$\mathbf{J} = -\mathbf{a}_y J = \mathbf{a}_y \rho(y) u(y), \tag{5–8}$$

where the charge density $\rho(y)$ is a negative quantity. The velocity $\mathbf{u} = \mathbf{a}_y u(y)$ is related to the electric field intensity $\mathbf{E}(y) = \mathbf{a}_y E(y)$ by Newton's law of motion:

$$m\frac{du(y)}{dt} = -eE(y) = e\frac{dV(y)}{dy}, \tag{5–9}$$

where $m = 9.11 \times 10^{-31}$ (kg) and $-e = -1.60 \times 10^{-19}$ (C) are the mass and charge, respectively, of an electron. Noting that

$$\begin{aligned} m\frac{du}{dt} &= m\frac{du}{dy}\frac{dy}{dt} = mu\frac{du}{dy} \\ &= \frac{d}{dy}\left(\frac{1}{2}mu^2\right), \end{aligned}$$

we can rewrite Eq. (5–9) as

$$\frac{d}{dy}\left(\frac{1}{2}mu^2\right) = e\frac{dV}{dy}. \tag{5–10}$$

Integration of Eq. (5–10) gives

$$\tfrac{1}{2}mu^2 = eV, \tag{5–11}$$

where the constant of integration has been set to zero because at $y = 0$, $u(0) = V(0) = 0$. From Eq. (5–11) we obtain

$$u = \left(\frac{2e}{m}V\right)^{1/2}. \tag{5–12}$$

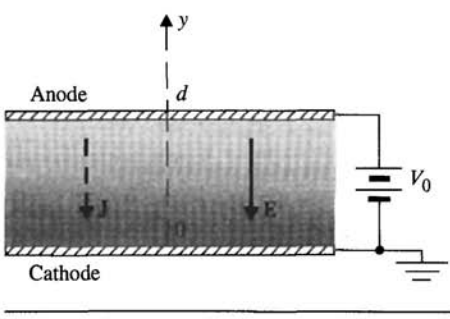

FIGURE 5–2
Space-charge-limited vacuum diode (Example 5–1).

In order to find $V(y)$ in the interelectrode region we must solve Poisson's equation with ρ expressed in terms of $V(y)$ from Eq. (5–8):

$$\rho = -\frac{J}{u} = -J\sqrt{\frac{m}{2e}}\,V^{-1/2}. \tag{5–13}$$

We have, from Eq. (4–6),

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{-1/2}. \tag{5–14}$$

Equation (5–14) can be integrated if both sides are first multiplied by $2\,dV/dy$. The result is

$$\left(\frac{dV}{dy}\right)^2 = \frac{4J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{1/2} + c. \tag{5–15}$$

At $y = 0$, $V = 0$, and $dV/dy = 0$ from Eq. (5–7), so $c = 0$. Equation (5–15) becomes

$$V^{-1/4}\,dV = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} dy. \tag{5–16}$$

Integrating the left side of Eq. (5–16) from $V = 0$ to V_0 and the right side from $y = 0$ to d, we obtain

$$\frac{4}{3}\,V_0^{3/4} = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} d,$$

or

$$J = \frac{4\epsilon_0}{9d^2}\sqrt{\frac{2e}{m}}\,V_0^{3/2} \qquad (\mathrm{A/m^2}). \tag{5–17}$$

Equation (5–17) states that the convection current density in a space-charge limited vacuum diode is proportional to the three-halves power of the potential difference between the anode and the cathode. This nonlinear relation is known as the ***Child-Langmuir law.*** ■

In the case of conduction currents there may be more than one kind of charge carriers (electrons, holes, and ions) drifting with different velocities. Equation (5–3) should be generalized to read

$$\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\mathrm{A/m^2}). \tag{5–18}$$

As indicated in Section 5–1, conduction currents are the result of the drift motion of charge carriers under the influence of an applied electric field. The atoms remain neutral ($\rho = 0$). It can be justified analytically that for most conducting materials the average drift velocity is directly proportional to the electric field intensity. For metallic conductors we write

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\mathrm{m/s}), \tag{5–19}$$

where μ_e is the electron ***mobility*** measured in $(\mathrm{m^2/V\cdot s})$. The electron mobility for copper is 3.2×10^{-3} $(\mathrm{m^2/V\cdot s})$. It is 1.4×10^{-4} $(\mathrm{m^2/V\cdot s})$ for aluminum and 5.2×10^{-3}

(m^2/V·s) for silver. From Eqs. (5–3) and (5–19) we have

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E}, \tag{5–20}$$

where $\rho_e = -Ne$ is the charge density of the drifting electrons and is a negative quantity. Equation (5–20) can be rewritten as

$$\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2),} \tag{5–21}$$

where the proportionality constant, $\sigma = -\rho_e \mu_e$, is a macroscopic constitutive parameter of the medium called ***conductivity***.

For semiconductors, conductivity depends on the concentration and mobility of both electrons and holes:

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h, \tag{5–22}$$

where the subscript h denotes hole. In general, $\mu_e \neq \mu_h$. For germanium, typical values are $\mu_e = 0.38$, $\mu_h = 0.18$; for silicon, $\mu_e = 0.12$, $\mu_h = 0.03$ (m^2/V·s).

Equation (5–21) is a constitutive relation of a conducting medium. Isotropic materials for which the linear relation Eq. (5–21) holds are called ***ohmic media***. The unit for σ is ampere per volt-meter (A/V·m) or siemens per meter (S/m). Copper, the most commonly used conductor, has a conductivity 5.80×10^7 (S/m). On the other hand, the conductivity of germanium is around 2.2 (S/m), and that of silicon is 1.6×10^{-3} (S/m). The conductivity of semiconductors is highly dependent of (increases with) temperature. Hard rubber, a good insulator, has a conductivity of only 10^{-15} (S/m). Appendix B–4 lists the conductivities of some other frequently used materials. However, note that, unlike the dielectric constant, the conductivity of materials varies over an extremely wide range. The reciprocal of conductivity is called ***resistivity***, in ohm-meters (Ω·m). We prefer to use conductivity; there is really no compelling need to use both conductivity and resistivity.

We recall ***Ohm's law*** from circuit theory that the voltage V_{12} across a resistance R, in which a current I flows from point 1 to point 2, is equal to RI; that is,

$$V_{12} = RI. \tag{5–23}$$

Here R is usually a piece of conducting material of a given length; V_{12} is the voltage between two terminals 1 and 2; and I is the total current flowing from terminal 1 to terminal 2 through a finite cross section.

Equation (5–23) is *not* a point relation. Although there is little resemblance between Eq. (5–21) and Eq. (5–23), the former is generally referred to as the ***point form of Ohm's law***. It holds at all points in space, and σ can be a function of space coordinates.

Let us use the point form of Ohm's law to derive the voltage-current relationship of a piece of homogeneous material of conductivity σ, length ℓ, and uniform cross section S, as shown in Fig. 5–3. Within the conducting material, $\mathbf{J} = \sigma\mathbf{E}$, where both $\mathbf{J}$ and $\mathbf{E}$ are in the direction of current flow. The potential difference or voltage

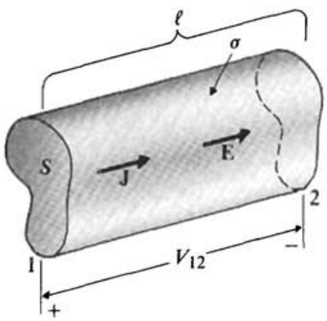

FIGURE 5–3
Homogeneous conductor with a constant cross section.

between terminals 1 and 2 is[†]

$$V_{12} = E\ell$$

or

$$E = \frac{V_{12}}{\ell}. \tag{5–24}$$

The total current is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS$$

or

$$J = \frac{I}{S}. \tag{5–25}$$

Using Eqs. (5–24) and (5–25) in Eq. (5–21), we obtain

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

or

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI, \tag{5–26}$$

which is the same as Eq. (5–23). From Eq. (5–26) we have the formula for the ***resistance*** of a straight piece of homogeneous material of a uniform cross section for steady current (d.c.):

$$\boxed{R = \frac{\ell}{\sigma S} \qquad (\Omega).} \tag{5–27}$$

We could have started with Eq. (5–23) as the experimental Ohm's law and applied it to a homogeneous conductor of length ℓ and uniform cross-section S. Using the formula in Eq. (5–27), we could derive the point relationship in Eq. (5–21).

[†] We will discuss the significance of V_{12} and E more in detail in Section 5–3.

EXAMPLE 5–2 Determine the d-c resistance of 1-(km) of wire having a 1-(mm) radius (a) if the wire is made of copper, and (b) if the wire is made of aluminum.

Solution Since we are dealing with conductors of a uniform cross section, Eq. (5–27) applies.

a) For copper wire, $\sigma_{cu} = 5.80 \times 10^7$ (S/m):

$$\ell = 10^3 \text{ (m)}, \qquad S = \pi(10^{-3})^2 = 10^{-6}\pi \quad (\text{m}^2).$$

We have

$$R_{cu} = \frac{\ell}{\sigma_{cu} S} = \frac{10^3}{5.80 \times 10^7 \times 10^{-6}\pi} = 5.49 \quad (\Omega).$$

b) For aluminum wire, $\sigma_{al} = 3.54 \times 10^7$ (S/m):

$$R_{al} = \frac{\ell}{\sigma_{al} S} = \frac{\sigma_{cu}}{\sigma_{al}} R_{cu} = \frac{5.80}{3.54} \times 5.49 = 8.99 \quad (\Omega).$$

The ***conductance***, G, or the reciprocal of resistance, is useful in combining resistances in parallel. The unit for conductance is (Ω^{-1}), or siemens (S).

$$G = \frac{1}{R} = \sigma \frac{S}{\ell} \quad \text{(S)}. \tag{5–28}$$

From circuit theory we know the following:

a) When resistances R_1 and R_2 are connected in series (same current), the total resistance R is

$$\boxed{R_{sr} = R_1 + R_2.} \tag{5–29}$$

b) When resistances R_1 and R_2 are connected in parallel (same voltage), we have

$$\boxed{\frac{1}{R_{\|}} = \frac{1}{R_1} + \frac{1}{R_2}} \tag{5–30a}$$

or

$$\boxed{G_{\|} = G_1 + G_2.} \tag{5–30b}$$

5–3 Electromotive Force and Kirchhoff's Voltage Law

In Section 3–2 we pointed out that static electric field is conservative and that the scalar line integral of static electric intensity around any closed path is zero; that is,

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0. \tag{5–31}$$