PEARSON NEW INTERNATIONAL EDITION

Classical Mechanics

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CLASSICAL MECHANICS

Effects due to the Coriolis terms also appear in atomic physics. Thus, two types of motion may occur simultaneously in polyatomic molecules: The molecule *rotates* as a rigid whole, and the atoms *vibrate* about their equilibrium positions. As a result of the vibrations, the atoms are in motion relative to the rotating coordinate system of the molecule. The Coriolis term will then be different from zero and will cause the atoms to move in a direction perpendicular to the original oscillations. Perturbations in molecular spectra due to Coriolis effects thus appear as interactions between the rotational and vibrational motions of the molecule.

DERIVATIONS

1. Prove that matrix multiplication is associative. Show that the product of two orthogonal matrices is also orthogonal.

2. Prove the following properties of the transposed and adjoint matrices:

$$\widetilde{\mathbf{AB}} = \tilde{\mathbf{B}}\tilde{\mathbf{A}},$$
$$(\mathbf{AB})^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger}.$$

3. Show that the trace of a matrix is invariant under any similarity transformation. Show also that the antisymmetry property of a matrix is preserved under an orthogonal similarity transformation.

4. **(a)** By examining the eigenvalues of an antisymmetric 3×3 real matrix $\mathbf{A}$, show that $\mathbf{1} \pm \mathbf{A}$ is nonsingular.

   (b) Show then that under the same conditions the matrix

$$\mathbf{B} = (\mathbf{1} + \mathbf{A})(\mathbf{1} - \mathbf{A})^{-1}$$

   is orthogonal.

5. Obtain the matrix elements of the general rotation matrix in terms of the Euler angles, Eq. (4.46), by performing the multiplications of the successive component rotation matrices. Verify directly that the matrix elements obey the orthogonality conditions.

6. The body set of axes can be related to the space set in terms of Euler's angles by the following set of rotations:

   (a) Rotation about the x axis by an angle θ.

   (b) Rotation about the z' axis by an angle ψ.

   (c) Rotation about the *old* z axis by an angle ϕ.

   Show that this sequence leads to the same elements of the matrix of transformation as the sequence of rotations given in the book. [*Hint:* It is not necessary to carry out the explicit multiplication of the rotation matrices.]

7. If $\mathbf{A}$ is the matrix of a rotation through 180° about any axis, show that if

$$\mathbf{P}_{\pm} = \tfrac{1}{2}(\mathbf{1} \pm \mathbf{A}),$$

then $\mathbf{P}_{\pm}^2 = \mathbf{P}_{\pm}$. Obtain the elements of $\mathbf{P}_{\pm}$ in any suitable system, and find a geometric interpretation of the operation $\mathbf{P}_{+}$ and $\mathbf{P}_{-}$ on any vector $\mathbf{F}$.

8. **(a)** Show that the rotation matrix in the form of Eq. (4.47′) cannot be put in the form of the matrix of the inversion transformation **S**.

(b) Verify by direct multiplication that the matrix in Eq. (4.47′) is orthogonal.

9. Show that any rotation can be represented by successive reflection in two planes, both passing through the axis of rotation with the planar angle $\Phi/2$ between them.

10. If **B** is a square matrix and **A** is the exponential of **B**, defined by the infinite series expansion of the exponential,

$$\mathbf{A} \equiv e^{\mathbf{B}} = \mathbf{1} + \mathbf{B} + \frac{1}{2}\mathbf{B}^2 + \cdots + \frac{\mathbf{B}^n}{n!} + \cdots,$$

then prove the following properties:

(a) $e^{\mathbf{B}}e^{\mathbf{C}} = e^{\mathbf{B}+\mathbf{C}}$, providing **B** and **C** commute.

(b) $\mathbf{A}^{-1} = e^{-\mathbf{B}}$

(c) $e^{\mathbf{CBC}^{-1}} = \mathbf{CAC}^{-1}$

(d) **A** is orthogonal if **B** is antisymmetric.

11. Verify the relation

$$|-\mathbf{B}| = (-1)^n|\mathbf{B}|$$

for the determinant of an $n \times n$ matrix **B**.

12. In a set of axes where the z axis is the axis of rotation of a finite rotation, the rotation matrix is given by Eq. (4.43) with ϕ replaced by the angle of finite rotation Φ. Derive the rotation formula, Eq. (4.62), by transforming to an arbitrary coordinate system, expressing the orthogonal matrix of transformation in terms of the direction cosines of the axis of the finite rotation.

13. **(a)** Suppose two successive coordinate rotations through angles Φ_1 and Φ_2 are carried out, equivalent to a single rotation through an angle Φ. Show that Φ_1, Φ_2, and Φ can be considered as the sides of a spherical triangle with the angle opposite to Φ given by the angle between the two axes of rotation.

(b) Show that a rotation about any given axis can be obtained as the product of two successive rotations, each through 180°.

14. **(a)** Verify that the permutation symbol satisfies the following identity in terms of Kronecker delta symbols:

$$\epsilon_{ijp}\epsilon_{rmp} = \delta_{ir}\delta_{jm} - \delta_{im}\delta_{jr}.$$

(b) Show that

$$\epsilon_{ijp}\epsilon_{ijk} = 2\delta_{pk}.$$

15. Show that the components of the angular velocity along the space set of axes are given in terms of the Euler angles by

$$\omega_x = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi,$$
$$\omega_y = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi,$$
$$\omega_z = \dot{\psi}\cos\theta + \dot{\phi}.$$

16. Show that the Euler parameter e_0 has the equation of motion

$$-2\dot{e}_0 = e_1\omega_{x'} + e_2\omega_{y'} + e_3\omega_{z'},$$

where the prime denotes the body set of axes. Find the corresponding equations for the other three Euler parameters and for the complex Cayley–Klein parameters α and β.

17. Verify directly that the matrix generators of infinitesimal rotation, $\mathbf{M}_i$, as given by Eq. (4.79) obey the commutation relations

$$[\mathbf{M}_i, \mathbf{M}_j] = \epsilon_{ijk}\mathbf{M}_k.$$

18. **(a)** Find the vector equation describing the reflection of **r** in a plane whose unit normal is **n**.

(b) Show that if l_i, $i = 1, 2, 3$, are the direction cosines of **n**, then the matrix of transformation has the elements

$$A_{ij} = \delta_{ij} - 2l_i l_j,$$

and verify that **A** is an improper orthogonal matrix.

19. Figures 4.9 and 4.10 show that the order of finite rotations leads to different results. Use the notation that $\mathbf{A}(\alpha, \mathbf{1}_n)$ where **A** is a rotation in the direction of $\mathbf{1}_n$ through an angle α. Let n_1 and n_2 be two orthogonal directions.

(a) If **x** is the position vector of a point on a rigid body, which is then rotated by an angle θ around the origin, show that the new value of **x** is

$$\mathbf{x}' = (\mathbf{1}_n \cdot \mathbf{x})\mathbf{1}_n + [\mathbf{x} - \mathbf{1}_n(\mathbf{1}_n \cdot \mathbf{x})]\cos\theta - \mathbf{1}_n \times \mathbf{x}\sin\theta.$$

From this, obtain the formula for $\mathbf{A}(\pi/2, \mathbf{1}_n)$ and derive the two rotations in the figures.

(b) Discuss these two rotations. [*Hint:* The answer will involve a rotation by the angle $\frac{2}{3}\pi$ in a direction $(1/\sqrt{3})(1, 1, 1)$.]

20. Express the "rolling" constraint of a sphere on a plane surface in terms of the Euler angles. Show that the conditions are nonintegrable and that the constraint is therefore nonholonomic.

EXERCISES

21. A particle is thrown up vertically with initial speed v_0, reaches a maximum height and falls back to ground. Show that the Coriolis deflection when it again reaches the ground is opposite in direction, and four times greater in magnitude, than the Coriolis deflection when it is dropped at rest from the same maximum height.

22. A projectile is fired horizontally along Earth's surface. Show that to a first approximation the angular deviation from the direction of fire resulting from the Coriolis effect varies linearly with time at a rate

$$\omega \cos \theta,$$

where ω is the angular frequency of Earth's rotation and θ is the co-latitude, the direction of deviation being to the right in the northern hemisphere.

23. The Foucault pendulum experiment consists in setting a long pendulum in motion at a point on the surface of the rotating Earth with its momentum originally in the vertical plane containing the pendulum bob and the point of suspension. Show that the pendulum's subsequent motion may be described by saying that the plane of oscillation rotates uniformly $2\pi \cos \theta$ radians per day, where θ is the co-latitude. What is the direction of rotation? The approximation of small oscillations may be used, if desired.

24. A wagon wheel with spokes is mounted on a vertical axis so it is free to rotate in the horizontal plane. The wheel is rotating with an angular speed of $\omega = 3.0$ radian/s. A bug crawls out on one of the spokes of the wheel with a velocity of 0.5 cm/s holding on to the spoke with a coefficient of friction $\mu = 0.30$. How far can the bug crawl along the spoke before it starts to slip?

25. A carousel (counter-clockwise merry-go-round) starts from rest and accelerates at a constant angular accleration of 0.02 revolutions/s^2. A girl sitting on a bench on the platform 7.0 m from the center is holding a 3.0 kg ball. Calculate the magnitude and direction of the force she must exert to hold the ball 6.0 s after the carousel starts to move. Give the direction with respect to the line from the center of rotation to the girl.

CHAPTER 5

The Rigid Body Equations of Motion

Chapter 4 presents all the kinematical tools needed in the discussion of rigid body motion. In the Euler angles we have a set of three coordinates, defined rather unsymmetrically it is true, yet suitable for use as the generalized coordinates describing the orientation of the rigid body. In addition, the method of orthogonal transformations, and the associated matrix algebra, furnish a powerful and elegant technique for investigating the characteristics of rigid body motion. We have already had one application of the technique in deriving Eq. (4.86), the relation between the states of change of a vector as viewed in the space system and in the body system. These tools will now be applied to obtain the Euler dynamical equations of motion of the rigid body in their most convenient form. With the help of the equations of motion, some simple but highly important problems of rigid body motion can be discussed.

5.1 ■ ANGULAR MOMENTUM AND KINETIC ENERGY OF MOTION ABOUT A POINT

Chasles' theorem states that any general displacement of a rigid body can be represented by a translation plus a rotation. The theorem suggests that it ought to be possible to split the problem of rigid body motion into two separate phases, one concerned solely with the translational motion of the body, the other, with its rotational motion. Of course, if one point of the body is fixed, the separation is obvious, for then there is only a rotational motion about the fixed point, without any translation. But even for a general type of motion such a separation is often possible. The six coordinates needed to describe the motion have already been formed into two sets in accordance with such a division: the three Cartesian coordinates of a point fixed in the rigid body to describe the translational motion and, say, the three Euler angles for the motion about the point. If, further, the origin of the body system is chosen to be the center of mass, then by Eq. (1.28) the total angular momentum divides naturally into contributions from the translation of the center of mass and from the rotation about the center of mass. The former term will involve only the Cartesian coordinates of the center of mass, the latter only the angle coordinates. By Eq. (1.31), a similar division holds for the total kinetic energy T, which can be written in the form

$$T = \tfrac{1}{2}Mv^2 + T'(\phi, \theta, \psi),$$

as the sum of the kinetic energy of the entire body as if concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Often the potential energy can be similarly divided, each term involving only one of the coordinate sets, either the translational or rotational. Thus, the potential energy in a uniform gravitational field will depend only upon the Cartesian vertical coordinate of the center of gravity.* Or if the force on a body is due to a uniform magnetic field, **B**, acting on its magnetic dipole moment, **M**, then the potential is proportional to $\mathbf{M} \cdot \mathbf{B}$, which involves only the orientation of the body. Certainly, almost all problems soluble in practice will allow for such a separation. In such a case, the entire mechanical problem does indeed split into two. The Lagrangian, $L = T - V$, divides into two parts, one involving only the translational coordinates, the other only the angle coordinates. These two groups of coordinates will then be completely separated, and the translational and rotational problems can be solved independently of each other.

It is of obvious importance therefore to obtain expressions for the angular momentum and kinetic energy of the motion about some point fixed in the body. To do so, we will make abundant use of Eq. (4.86) linking derivatives relative to a coordinate system fixed at some point in the rigid body. It is intuitively obvious that the rotation angle of a rigid body displacement, as also the instantaneous angular velocity vector, is independent of the choice of origin of the body system of axes. The essence of the rigid body constraint is that all particles of the body move and rotate together. However, a formal proof is easily constructed.

Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the position vectors, relative to a fixed set of coordinates, of the origins of two sets of body coordinates (cf. Fig. 5.1). The difference vector is denoted by $\mathbf{R}$:

$$\mathbf{R}_2 = \mathbf{R}_1 + \mathbf{R}.$$

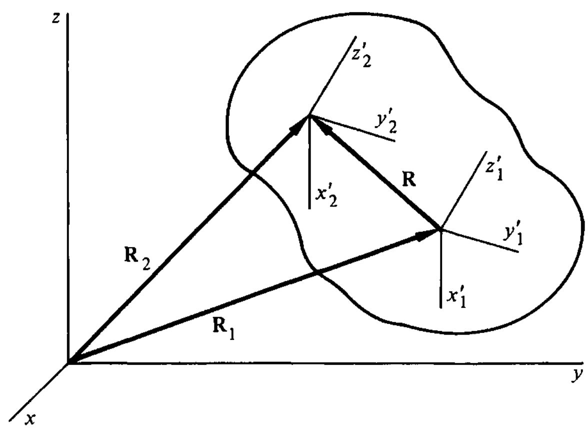

FIGURE 5.1 Vectorial relation between sets of rigid body coordinates with different origins.

*The center of gravity of course coincides with the center of mass in a uniform gravitational field.