Pearson New International Edition

Discrete-Time Signal Processing

Alan V. Oppenheim Ronald W. Schafer

Third Edition

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Example 8 3rd–Order IIR System

In this example, we consider a lowpass filter designed using an approximation method. The system function to be considered is

$$H(z) = \frac{0.05634(1 + z^{-1})(1 - 1.0166z^{-1} + z^{-2})}{(1 - 0.683z^{-1})(1 - 1.4461z^{-1} + 0.7957z^{-2})}, \tag{67}$$

and the system is specified to be stable. The zeros of this system function are at the following locations:

Radius	Angle
1	π rad
1	± 1.0376 rad (59.45°)

The poles are at the following locations:

Radius	Angle
0.683	0
0.892	± 0.6257 rad (35.85°)

The pole–zero plot for this system is shown in Figure 14. Figure 15 shows the log magnitude, phase, and group delay of the system. The effect of the zeros that are on the unit circle at $\omega = \pm 1.0376$ and π is clearly evident. However, the poles are placed so that, rather than peaking for frequencies close to their angles, the total log magnitude remains close to 0 dB over a band from $\omega = 0$ to $\omega = 0.2\pi$ (and, by symmetry, from $\omega = 1.8\pi$ to $\omega = 2\pi$), and then it drops abruptly and remains below -25 dB from about $\omega = 0.3\pi$ to 1.7π. As suggested by this example, useful approximations to frequency-selective filter responses can be achieved using poles to build up the magnitude response and zeros to suppress it.

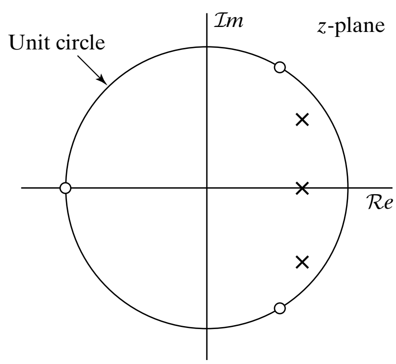

Figure 14 Pole–zero plot for the lowpass filter of Example 8.

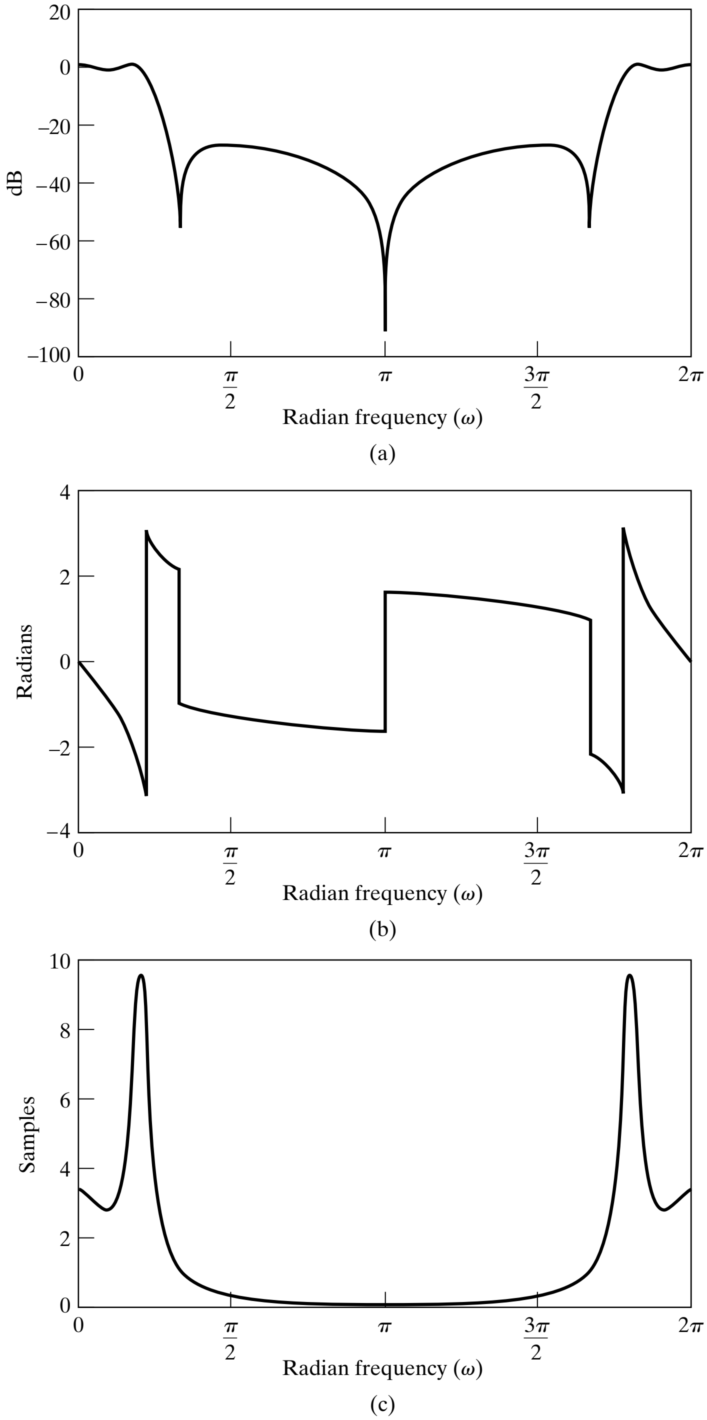

Figure 15 Frequency response for the lowpass filter of Example 8. (a) Log magnitude. (b) Phase. (c) Group delay.

In this example, we see both types of discontinuities in the plotted phase curve. At $\omega \approx 0.22\pi$, there is a discontinuity of 2π owing to the use of the principal value in plotting. At $\omega = \pm 1.0376$ and $\omega = \pi$, the discontinuities of π are due to the zeros on the unit circle.

4 RELATIONSHIP BETWEEN MAGNITUDE AND PHASE

In general, knowledge about the magnitude of the frequency response of an LTI system provides no information about the phase, and vice versa. However, for systems described by linear constant-coefficient difference equations, i.e., rational system functions, there is some constraint between magnitude and phase. In particular, as we discuss in this section, if the magnitude of the frequency response and the number of poles and zeros are known, then there are only a finite number of choices for the associated phase. Similarly, if the number of poles and zeros and the phase are known, then, to within a scale factor, there are only a finite number of choices for the magnitude. Furthermore, under a constraint referred to as minimum phase, the frequency-response magnitude specifies the phase uniquely, and the frequency-response phase specifies the magnitude to within a scale factor.

To explore the possible choices of system function, given the square of the magnitude of the system frequency response, we consider $|H(e^{j\omega})|^2$ expressed as

$$\begin{aligned} |H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) \\ &= H(z)H^*(1/z^*)|_{z=e^{j\omega}}. \end{aligned} \tag{68}$$

Restricting the system function $H(z)$ to be rational in the form of Eq. (21), i.e.,

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}, \tag{69}$$

we see that $H^*(1/z^*)$ in Eq. (68) is

$$H^*\left(\frac{1}{z^*}\right) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k^* z)}, \tag{70}$$

wherein we have assumed that a_0 and b_0 are real. Therefore, Eq. (68) states that the square of the magnitude of the frequency response is the evaluation on the unit circle of the z-transform

$$C(z) = H(z)H^*(1/z^*) \tag{71}$$

$$= \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}. \tag{72}$$

If we know $|H(e^{j\omega})|^2$ expressed as a function of $e^{j\omega}$, then by replacing $e^{j\omega}$ by z, we can construct $C(z)$. From $C(z)$, we would like to infer as much as possible about $H(z)$. We first note that for each pole d_k of $H(z)$, there is a pole of $C(z)$ at d_k and $(d_k^*)^{-1}$. Similarly, for each zero c_k of $H(z)$, there is a zero of $C(z)$ at c_k and $(c_k^*)^{-1}$. Consequently, the poles and zeros of $C(z)$ occur in conjugate reciprocal pairs, with one element of each pair associated with $H(z)$ and one element of each pair associated with $H^*(1/z^*)$. Furthermore, if one element of each pair is inside the unit circle, then the other (i.e., the conjugate reciprocal) will be outside the unit circle. The only other alternative is for both to be on the unit circle in the same location.

If $H(z)$ is assumed to correspond to a causal, stable system, then all its poles must lie inside the unit circle. With this constraint, the poles of $H(z)$ can be identified from the poles of $C(z)$. However, with this constraint alone, the zeros of $H(z)$ cannot be uniquely identified from the zeros of $C(z)$. This can be seen from the following example.

Example 9 Different Systems with the Same C(z)

Consider two different stable systems with system functions

$$H_1(z) = \frac{2(1 - z^{-1})(1 + 0.5z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} \tag{73}$$

and

$$H_2(z) = \frac{(1 - z^{-1})(1 + 2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}. \tag{74}$$

The pole–zero plots for these systems are shown in Figures 16(a) and 16(b), respectively. The two systems have identical pole locations and both have a zero at $z = 1$ but differ in the location of the second zero.

Now,

$$C_1(z) = H_1(z)H_1^*(1/z^*)$$
$$= \frac{2(1 - z^{-1})(1 + 0.5z^{-1})2(1 - z)(1 + 0.5z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)} \tag{75}$$

and

$$C_2(z) = H_2(z)H_2^*(1/z^*)$$
$$= \frac{(1 - z^{-1})(1 + 2z^{-1})(1 - z)(1 + 2z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)}. \tag{76}$$

Using the fact that

$$4(1 + 0.5z^{-1})(1 + 0.5z) = (1 + 2z^{-1})(1 + 2z), \tag{77}$$

we see that $C_1(z) = C_2(z)$. The pole–zero plot for $C_1(z)$ and $C_2(z)$, which are identical, is shown in Figure 16(c).

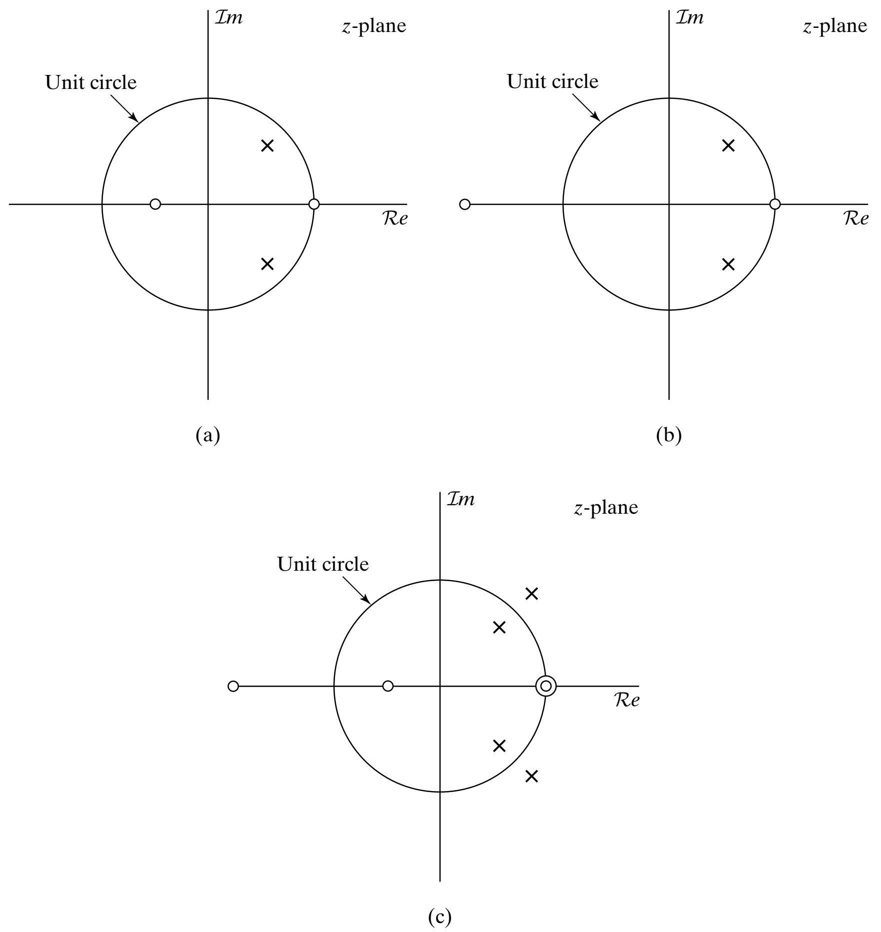

Figure 16 Pole–zero plots for two system functions and their common magnitude-squared function. (a) $H_1(z)$. (b) $H_2(z)$. (c) $C_1(z)$, $C_2(z)$.

The system functions $H_1(z)$ and $H_2(z)$ in Example 9 differ only in the location of one of the zeros. In the example, the factor $2(1 + 0.5z^{-1}) = (z^{-1} + 2)$ contributes the same to the square of the magnitude of the frequency response as the factor $(1 + 2z^{-1})$, and consequently, $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$ are equal. However, the phase functions for these two frequency responses are different.

Example 10 Determination of $H(z)$ from $C(z)$

Suppose we are given the pole–zero plot for $C(z)$ in Figure 17 and want to determine the poles and zeros to associate with $H(z)$. The conjugate reciprocal pairs of poles and zeros for which one element of each is associated with $H(z)$ and one with $H^*(1/z^*)$ are as follows:

$$\begin{aligned}
\text{Pole pair 1}: &\quad (p_1, p_4)\\
\text{Pole pair 2}: &\quad (p_2, p_5)\\
\text{Pole pair 3}: &\quad (p_3, p_6)\\
\text{Zero pair 1}: &\quad (z_1, z_4)\\
\text{Zero pair 2}: &\quad (z_2, z_5)\\
\text{Zero pair 3}: &\quad (z_3, z_6)
\end{aligned}$$

Knowing that $H(z)$ corresponds to a stable, causal system, we must choose the poles from each pair that are inside the unit circle, i.e., p_1, p_2, and p_3. No such constraint is imposed on the zeros. However, if we assume that the coefficients a_k and b_k are real in Eqs. (19) and (20), the zeros (and poles) either are real or occur in complex conjugate pairs. Consequently, the zeros to associate with $H(z)$ are

$$z_3 \quad \text{or} \quad z_6$$

and

$$(z_1, z_2) \quad \text{or} \quad (z_4, z_5).$$

Therefore, there are a total of four different stable, causal systems with three poles and three zeros for which the pole–zero plot of $C(z)$ is that shown in Figure 17 and, equivalently, for which the frequency-response magnitude is the same. If we had not assumed that the coefficients a_k and b_k were real, the number of choices would be greater. Furthermore, if the number of poles and zeros of $H(z)$ were not restricted, the number of choices for $H(z)$ would be unlimited. To see this, assume that $H(z)$ has a factor of the form

$$\frac{z^{-1} - a^*}{1 - az^{-1}},$$

i.e.,

$$H(z) = H_1(z)\frac{z^{-1} - a^*}{1 - az^{-1}}. \tag{78}$$

Factors of this form represent *all-pass factors,* since they have unity magnitude on the unit circle; they are discussed in more detail in Section 5. It is easily verified that for $H(z)$ in Eq. (78),

$$C(z) = H(z)H^*(1/z^*) = H_1(z)H_1^*(1/z^*); \tag{79}$$

i.e., all-pass factors cancel in $C(z)$ and therefore would not be identifiable from the pole–zero plot of $C(z)$. Consequently, if the number of poles and zeros of $H(z)$ is unspecified, then, given $C(z)$, any choice for $H(z)$ can be cascaded with an arbitrary number of all-pass factors with poles inside the unit circle (i.e., $|a| < 1$).