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Fundamentals of Complex Analysis Engineering, Science, and Mathematics

Edward B. Saff Arthur David Snider

Third Edition

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In fact, there is no reason to insist that the parameter be called "t"; the formula

$$z(x) = x \quad (1 \le x \le 8)$$

is quite satisfactory.

(b) The point set is $z = 2 + iy$, $-2 \le y \le 2$, so simply take

$$z(y) = 2 + iy \quad (-2 \le y \le 2).$$

(c) Given any two distinct points z_1 and z_2, every point on the line segment joining z_1 and z_2 is of the form $z_1 + t\,(z_2 - z_1)$, where $0 \le t \le 1$ (see Prob. 18 in Exercises 1.3). Therefore, the given curve constitutes the range of

$$z(t) = -2 - 3i + t(7 + 9i) \quad (0 \le t \le 1).$$

(d) In Section 1.4 it was shown that any point on the unit circle centered at the origin can be written in the form $e^{i\theta} = \cos\theta + i\sin\theta$ for $0 \le \theta < 2\pi$; therefore, an admissible parametrization for this smooth closed curve is constructed by interpreting θ as the parameter: $z_0(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$ (notice that the endpoints are joined properly). To parametrize the given circle (d) we simply shift the center and double the radius:

$$z(\theta) = 1 - i + 2e^{i\theta} \quad (0 \le \theta \le 2\pi).$$

Note that by suitably restricting the limits on θ we could generate a semicircle or any other circular arc.

(e) The parametrization of the graph of any function $y = f(x)$ is also easy; simply let x be the parameter and write $z(x) = x + if(x)$, and set the limits. This is an admissible parametrization as long as $f(x)$ is continuously differentiable. For the graph (e) we have

$$z(x) = x + ix^3 \quad (0 \le x \le 1).$$

The verification of conditions (i), (ii), and (iii) (or (iii$'$)) is immediate for each of these curves; thus these are admissible parametrizations. ■

Let us carry our analysis of curve sketching a little further. Suppose that the artist is to draw a smooth *arc* like that in Fig. 4.2 and is to abide by our ground rules (in particular it is illegal to retrace points). Then it is intuitively clear that the artist must either start at z_I and work toward z_{II}, or start at z_{II} and terminate at z_I. Either mode produces an ordering of the points along the curve (Fig. 4.3).

Thus we see that there are exactly two such "natural" orderings of the points of a smooth arc γ, and either one can be specified by declaring which endpoint of γ is the initial point. A smooth arc, together with a specific ordering of its points, is called a *directed smooth arc*. The ordering can be indicated by an arrow, as in Fig. 4.3.

The ordering that the artist generates while drawing γ is reflected in the parametrization function $z(t)$ that describes the pen's trajectory; specifically, the point $z\,(t_1)$

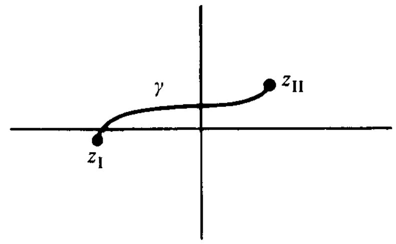

Figure 4.2 Smooth arc.

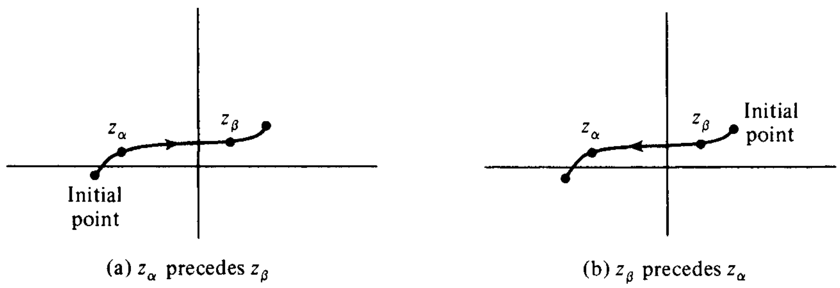

Figure 4.3 Directed smooth arcs.

will precede the point $z\,(t_2)$ whenever $t_1 < t_2$. Since there are only two possible (natural) orderings, *any* admissible parametrization must fall into one of two categories, according to the particular ordering it respects. In general, if $z = z(t)$, $a \le t \le b$, is an admissible parametrization consistent with one of the orderings, then $z = z(-t)$, $-b \le t \le -a$, always corresponds to the opposite ordering.

The situation is slightly more complicated if the artist is to draw a smooth *closed* curve. First an initial point must be selected; then the artist must choose one of the two directions in which to trace the curve (see Fig. 4.4). Having made these decisions, the artist has established the ordering of the points of γ. Now, however, there is one anomaly; the initial point both precedes and is preceded by every other point, since it also serves as the terminal point. Ignoring this schizophrenic pest, we shall say that the points of a smooth closed curve have been ordered when (i) a designation of the initial point is made and (ii) one of the two "directions of transit" from this point is selected. A smooth closed curve whose points have been ordered is called a *directed smooth closed curve*.

As in the case of smooth arcs, the parametrization of the trajectory of the artist's pen reflects the ordering generated in sketching a smooth closed curve. If this parametrization is given by $z = z(t)$, $a \le t \le b$, then (i) the initial point must be $z(a)$ and (ii) the point $z\,(t_1)$ precedes the point $z\,(t_2)$ whenever $a < t_1 < t_2 < b$. Any other admissible parametrization having the same initial point must reflect either the same or the opposite ordering.

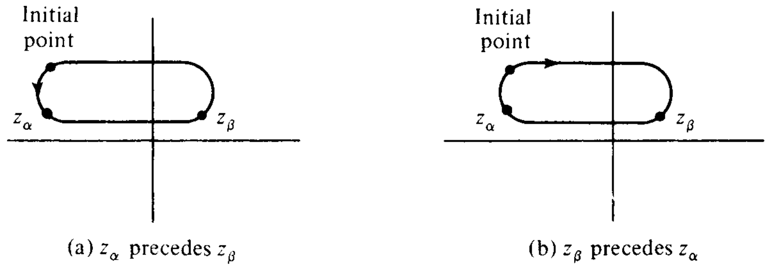

Figure 4.4 Directed smooth closed curves.

The phrase *directed smooth curve* will be used to mean either a directed smooth arc or a directed smooth closed curve.

Now we are ready to specify the more general kinds of curves that will be used in the theory of integration. They are formed by joining directed smooth curves together end-to-end; this allows self-intersections, cusps, and corners. In addition, it will be convenient to include single isolated points as members of this class. Let us explore the possibilities uncovered by the following definition.

Definition 2. A **contour** Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \ldots, n-1$. In this case one can write $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.

Notice that a single directed smooth curve is a contour with $n = 1$.

Speaking loosely, we can say that the contour Γ inherits a direction from its components γ_k: If z_1 and z_2 lie on the same directed smooth curve γ_k, they are ordered by the direction on γ_k, and if z_1 lies on γ_i while z_2 lies on γ_j, we say that z_1 precedes z_2 if $i < j$. This is ambiguous because of the possibility that a point of self-intersection, say z_1, would lie on two different smooth curves, and therefore we must indicate which "occurrence" of z_1 is meant when we say z_1 precedes z_2.

Figure 4.5 illustrates four elementary examples of contours formed by joining directed smooth curves. In Fig. 4.5(d), if z_α is regarded as a point of γ_1 it precedes z_β, but regarded as a point of γ_3, it is preceded by z_β.

Figs. 4.6, 4.7, and 4.8 depict three interesting contours that will be employed when we study examples of contour integration in Chapter 6. Note that in Fig. 4.7 we retrace entire segments in the course of tracing the contour.

A parametrization of a contour is simply a "piecing together" of admissible parametrizations of its smooth-curve components. We will never have need to carry this out explicitly, because in practice we always break up a contour into its smooth-curve components. However the theory is much easier to express in terms of contour parametrizations, so let us spell it out once and for all. One says that $z = z(t)$, $a \le t \le b$, is a

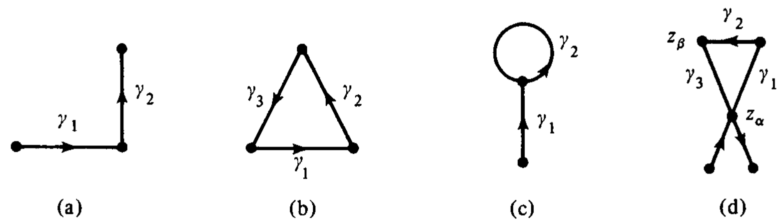

Figure 4.5 Examples of contours.

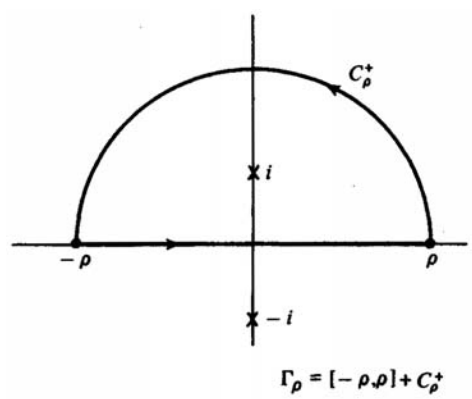

Figure 4.6 Semicircular contour.

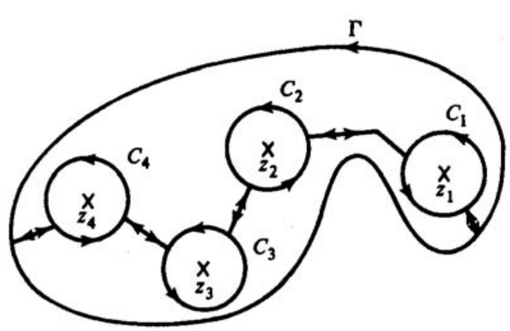

Figure 4.7 Contour with intrusions.

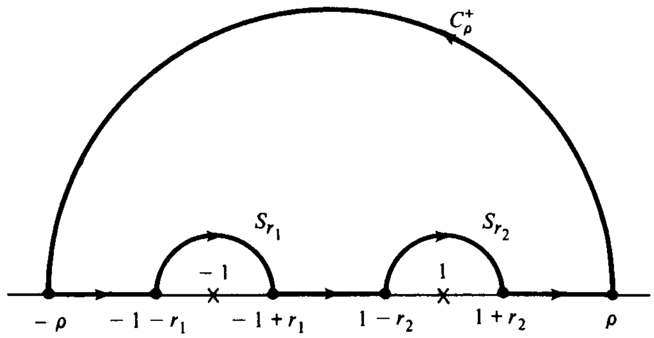

Figure 4.8 Contour with indentations.

parametrization of the contour $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ if there is a subdivision of $[a, b]$ into n subintervals $[\tau_0, \tau_1], [\tau_1, \tau_2], \ldots, [\tau_{n-1}, \tau_n]$, where

$$a = \tau_0 < \tau_1 < \cdots < \tau_{n-1} < \tau_n = b,$$

such that on each subinterval $[\tau_{k-1}, \tau_k]$ the function $z(t)$ is an admissible parametrization of the smooth curve γ_k, consistent with the direction on γ_k. Since the endpoints of consecutive γ_k's are properly connected, $z(t)$ must be continuous on $[a, b]$. However $z'(t)$ may have jump discontinuities at the points τ_k.

The contour parametrization of a point is simply a constant function.

When we have admissible parametrizations of the components γ_k of a contour Γ, we can piece these together to get a contour parametrization for Γ by simply rescaling and shifting the parameter intervals for t. The technique is amply illustrated by the following example. (The general case is discussed in Prob. 6.)

Example 2

Parametrize the contour in Fig. 4.9, for t in the interval $0 \le t \le 1$.

Solution. We have already seen how to parametrize straight lines. The following functions are admissible parametrizations for γ_1, γ_2, and γ_3, consistent with their

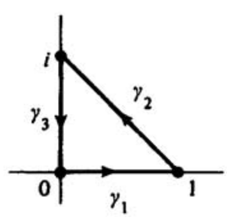

Figure 4.9 Contour for Example 2.

directions:

$$\gamma_1: \quad z_1(t) = t \qquad (0 \le t \le 1),$$

$$\gamma_2: \quad z_2(t) = 1 + t(i-1) \quad (0 \le t \le 1),$$

$$\gamma_3: \quad z_3(t) = i - ti \qquad (0 \le t \le 1).$$

Now we rescale so that γ_1 is traced as t varies between 0 and $\frac{1}{3}$, γ_2 is traced for $\frac{1}{3} \le t \le \frac{2}{3}$, and γ_3 is traced for $\frac{2}{3} \le t \le 1$. This is simply a matter of shifting and stretching the variable t.

For γ_1, observe that the range of the function $z_1(t) = t$, $0 \le t \le 1$, is the same as the range of $z_I(t) = 3t$, $0 \le t \le \frac{1}{3}$, and that $z_I(t)$ is an admissible parametrization corresponding to the same ordering. The curve γ_2 is the range of $z_2(t) = 1 + t(i-1)$, $0 \le t \le 1$, and this is the same as the range of $z_{II}(t) = 1 + 3(t - \frac{1}{3})(i-1)$, $\frac{1}{3} \le t \le \frac{2}{3}$, again preserving admissibility and ordering. Handling $z_3(t)$ similarly, we find

$$z(t) = \begin{cases} 3t & \left(0 \le t \le \frac{1}{3}\right), \\ 1 + 3\left(t - \frac{1}{3}\right)(i-1) & \left(\frac{1}{3} \le t \le \frac{2}{3}\right), \\ i - 3\left(t - \frac{2}{3}\right)i & \left(\frac{2}{3} \le t \le 1\right). \quad \blacksquare \end{cases}$$

The (undirected) point set underlying a contour is known as a *piecewise smooth curve*. We shall use the symbol Γ ambiguously to refer to both the contour and its underlying curve, allowing the context to provide the proper interpretation.

Much of the terminology of directed smooth curves is readily applied to contours. The initial point of Γ is the initial point of γ_1, and its terminal point is the terminal point of γ_n; therefore Γ can be regarded as a path connecting these points. If the directions on all the components of Γ are reversed and the components are taken in the opposite order, the resulting contour is called the *opposite contour* and is denoted by $-\Gamma$ (see Fig. 4.10). Notice that if $z = z(t)$, $a \le t \le b$, is a parametrization of Γ, then $z = z(-t)$, $-b \le t \le -a$, parametrizes $-\Gamma$.

Γ is said to be a *closed contour* or a *loop* if its initial and terminal points coincide. A *simple closed contour* is a closed contour with no multiple points other than its

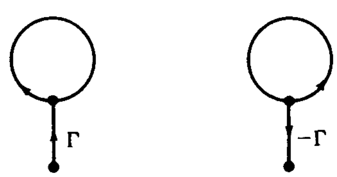

Figure 4.10 Oppositely oriented contours.