PEARSON NEW INTERNATIONAL EDITION

Elementary Linear Algebra with Applications

Bernard Kolman David Hill

Ninth Edition

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with Applications

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10. Does the set

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

span M_{22}?

11. Find a set of vectors spanning the solution space of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

12. Find a set of vectors spanning the null space of

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 6 & -2 \\ -2 & 1 & 2 & 2 \\ 0 & -2 & -4 & 0 \end{bmatrix}.$$

13. The set W of all 2×2 matrices A with trace equal to zero is a subspace of M_{22}. Let

$$S = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Show that span $S = W$.

14. The set W of all 3×3 matrices A with trace equal to zero is a subspace of M_{33}. (See Exercise 11 in Section 4.3.) Determine a subset S of W so that span $S = W$.

15. The set W of all 3×3 matrices of the form

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{bmatrix}$$

is a subspace of M_{33}. (See Exercise 12 in Section 4.3.) Determine a subset S of W so that span $S = W$.

16. Let T be the set of all matrices of the form $AB - BA$, where A and B are $n \times n$ matrices. Show that span T is not M_{nn}. (*Hint*: Use properties of trace.)

17. Determine whether your software has a command for finding the null space (see Example 10 in Section 4.3) of a matrix A. If it does, use it on the matrix A in Example 10 and compare the command's output with the results in Example 10. To experiment further, use Exercises 11 and 12.

4.5 Linear Independence

In Section 4.4 we developed the notion of the span of a set of vectors together with spanning sets of a vector space or subspace. Spanning sets S provide vectors so that any vector in the space can be expressed as a linear combination of the members of S. We remarked that a vector space can have many different spanning sets and that spanning sets for the same space need not have the same number of vectors. We illustrate this in Example 1.

EXAMPLE 1

In Example 5 of Section 4.3 we showed that the set W of all vectors of the form

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix},$$

where a and b are any real numbers, is a subspace of R^3. Each of the following sets is a spanning set for W (verify):

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\} \qquad S_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

■

We observe that the set S_3 is a more "efficient" spanning set, since each vector of W is a linear combination of two vectors, compared with three vectors when using S_1 and four vectors when using S_2. If we can determine a spanning set for a vector space V that is minimal, in the sense that it contains the fewest number of vectors, then we have an efficient way to describe every vector in V.

In Example 1, since the vectors in S_3 span W and S_3 is a subset of S_1 and S_2, it follows that the vector

$$\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

in S_1 must be a linear combination of the vectors in S_3, and similarly, both vectors

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

in S_2 must be linear combinations of the vectors in S_3. Observe that

$$3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix},$$

$$0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

In addition, for set S_1 we observe that

$$3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and for set S_2 we observe that

$$0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that if span $S = V$, and there is a linear combination of the vectors in S with coefficients not all zero that gives the zero vector, then some subset of S is also a spanning set for V.

Remark The preceding discussion motivates the next definition. In the preceding discussion, which is based on Example 1, we used the observation that S_3 was a subset of S_1 and of S_2. However, that observation is a special case which need not apply when comparing two spanning sets for a vector space.

DEFINITION 4.9

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly dependent** if there exist constants $a_1, a_2, \ldots, a_k$, not all zero, such that

$$\sum_{j=1}^{k} a_j \mathbf{v}_j = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}. \tag{1}$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if, whenever $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$,

$$a_1 = a_2 = \cdots = a_k = 0.$$

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then we also say that the set S is **linearly dependent** or **linearly independent** if the vectors have the corresponding property.

It should be emphasized that for any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, Equation (1) always holds if we choose all the scalars $a_1, a_2, \ldots, a_k$ equal to zero. The important point in this definition is whether it is possible to satisfy (1) with at least one of the scalars different from zero.

Remark Definition 4.9 is stated for a finite set of vectors, but it also applies to an infinite set S of a vector space, using corresponding notation for infinite sums.

Remark We connect Definition 4.9 to "efficient" spanning sets in Section 4.6.

To determine whether a set of vectors is linearly independent or linearly dependent, we use Equation (1). Regardless of the form of the vectors, Equation (1) yields a homogeneous linear system of equations. It is always consistent, since $a_1 = a_2 = \cdots = a_k = 0$ is a solution. However, the main idea from Definition 4.9 is whether there is a nontrivial solution.

EXAMPLE 2

Determine whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

are linearly independent.

Solution

Forming Equation (1),

$$a_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

we obtain the homogeneous system (verify)

$$\begin{aligned} 3a_1 + a_2 - a_3 &= 0 \\ 2a_1 + 2a_2 + 2a_3 &= 0 \\ a_1 - a_3 &= 0. \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{rrr|r} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right],$$

whose reduced row echelon form is (verify)

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus there is a nontrivial solution

$$\begin{bmatrix} k \\ -2k \\ k \end{bmatrix}, \quad k \neq 0 \text{ (verify)},$$

so the vectors are linearly dependent. ■

EXAMPLE 3 Are the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 1 & 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 1 & 1 & 1 & 3 \end{bmatrix}$ in R_4 linearly dependent or linearly independent?

Solution

We form Equation (1),

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0},$$

and solve for a_1, a_2, and a_3. The resulting homogeneous system is (verify)

$$\begin{aligned} a_1 + a_3 &= 0 \\ a_2 + a_3 &= 0 \\ a_1 + a_2 + a_3 &= 0 \\ 2a_1 + 2a_2 + 3a_3 &= 0. \end{aligned}$$

The corresponding augmented matrix is (verify)

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{array}\right],$$

and its reduced row echelon form is (verify)

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the only solution is the trivial solution $a_1 = a_2 = a_3 = 0$, so the vectors are linearly independent. ■

EXAMPLE 4 Are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$$

in M_{22} linearly independent?

Solution

We form Equation (1),

$$a_1 \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and solve for a_1, a_2, and a_3. Performing the scalar multiplications and adding the resulting matrices gives

$$\begin{bmatrix} 2a_1 + a_2 & a_1 + 2a_2 - 3a_3 \\ a_2 - 2a_3 & a_1 + a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Using the definition for equal matrices, we have the linear system

$$\begin{aligned} 2a_1 + a_2 \phantom{{}-3a_3} &= 0 \\ a_1 + 2a_2 - 3a_3 &= 0 \\ a_2 - 2a_3 &= 0 \\ a_1 \phantom{{}+2a_2} + a_3 &= 0. \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{rrr|r} 2 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right],$$

and its reduced row echelon form is (verify)

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus there is a nontrivial solution

$$\begin{bmatrix} -k \\ 2k \\ k \end{bmatrix}, \qquad k \neq 0 \text{ (verify)},$$

so the vectors are linearly dependent. ■

EXAMPLE 5

To find out whether the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ in R_3 are linearly dependent or linearly independent, we form Equation (1),

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0},$$

and solve for a_1, a_2, and a_3. Since $a_1 = a_2 = a_3 = 0$ (verify), we conclude that the given vectors are linearly independent. ■

EXAMPLE 6

Are the vectors $\mathbf{v}_1 = t^2 + t + 2$, $\mathbf{v}_2 = 2t^2 + t$, and $\mathbf{v}_3 = 3t^2 + 2t + 2$ in P_2 linearly dependent or linearly independent?

Solution

Forming Equation (1), we have (verify)

$$\begin{aligned} a_1 + 2a_2 + 3a_3 &= 0 \\ a_1 + a_2 + 2a_3 &= 0 \\ 2a_1 \phantom{{}+ 2a_2} + 2a_3 &= 0, \end{aligned}$$

which has infinitely many solutions (verify). A particular solution is $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, so

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

Hence the given vectors are linearly dependent. ■

EXAMPLE 7

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

in R^3. Is $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly dependent or linearly independent?

Solution

Setting up Equation (1), we are led to the homogeneous system

$$\begin{aligned} a_1 + a_2 - 3a_3 + 2a_4 &= 0 \\ 2a_1 - 2a_2 + 2a_3 \phantom{{}+ 2a_4} &= 0 \\ -a_1 + a_2 - a_3 \phantom{{}+ 2a_4} &= 0, \end{aligned}$$

of three equations in four unknowns. By Theorem 2.4, we are assured of the existence of a nontrivial solution. Hence S is linearly dependent. In fact, two of the infinitely many solutions are

$$\begin{aligned} &a_1 = 1, \quad a_2 = 2, \quad a_3 = 1, \quad a_4 = 0; \\ &a_1 = 1, \quad a_2 = 1, \quad a_3 = 0, \quad a_4 = -1. \end{aligned}$$ ■

EXAMPLE 8

Determine whether the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$