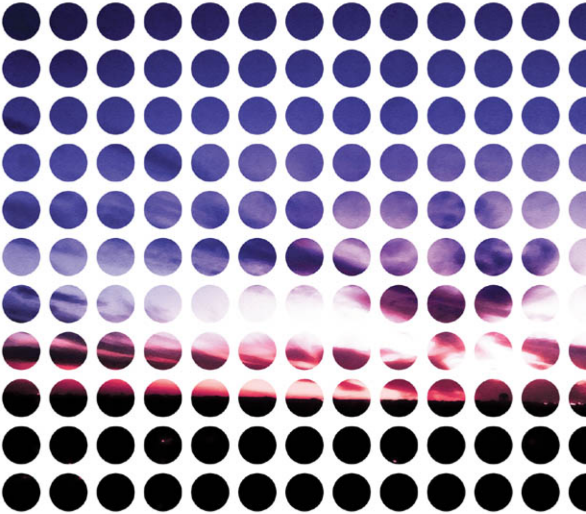

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Theorem 23.3. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces of a space X; let p be a point of $\bigcap A_\alpha$. We prove that the space $Y = \bigcup A_\alpha$ is connected. Suppose that $Y = C \cup D$ is a separation of Y. The point p is in one of the sets C or D; suppose $p \in C$. Since A_α is connected, it must lie entirely in either C or D, and it cannot lie in D because it contains the point p of C. Hence $A_\alpha \subset C$ for every α, so that $\bigcup A_\alpha \subset C$, contradicting the fact that D is nonempty. ∎

Theorem 23.4. *Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.*

Said differently: If B is formed by adjoining to the connected subspace A some or all of its limit points, then B is connected.

Proof. Let A be connected and let $A \subset B \subset \bar{A}$. Suppose that $B = C \cup D$ is a separation of B. By Lemma 23.2, the set A must lie entirely in C or in D; suppose that $A \subset C$. Then $\bar{A} \subset \bar{C}$; since $\bar{C}$ and D are disjoint, B cannot intersect D. This contradicts the fact that D is a nonempty subset of B. ∎

Theorem 23.5. *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \to Y$ be a continuous map; let X be connected. We wish to prove the image space $Z = f(X)$ is connected. Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map

$$g : X \to Z.$$

Suppose that $Z = A \cup B$ is a separation of Z into two disjoint nonempty sets open in Z. Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets whose union is X; they are open in X because g is continuous, and nonempty because g is surjective. Therefore, they form a separation of X, contradicting the assumption that X is connected. ∎

Theorem 23.6. *A finite cartesian product of connected spaces is connected.*

Proof. We prove the theorem first for the product of two connected spaces X and Y. This proof is easy to visualize. Choose a "base point" $a \times b$ in the product $X \times Y$. Note that the "horizontal slice" $X \times b$ is connected, being homeomorphic with X, and each "vertical slice" $x \times Y$ is connected, being homeomorphic with Y. As a result, each "T-shaped" space

$$T_x = (X \times b) \cup (x \times Y)$$

is connected, being the union of two connected spaces that have the point $x \times b$ in common. See Figure 23.2. Now form the union $\bigcup_{x \in X} T_x$ of all these T-shaped spaces.

This union is connected because it is the union of a collection of connected spaces that have the point $a \times b$ in common. Since this union equals $X \times Y$, the space $X \times Y$ is connected.

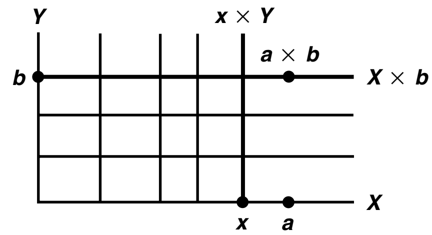

Figure 23.2

The proof for any finite product of connected spaces follows by induction, using the fact (easily proved) that $X_1 \times \cdots \times X_n$ is homeomorphic with $(X_1 \times \cdots \times X_{n-1}) \times X_n$. ∎

It is natural to ask whether this theorem extends to arbitrary products of connected spaces. The answer depends on which topology is used for the product, as the following examples show.

EXAMPLE 6. Consider the cartesian product $\mathbb{R}^\omega$ in the box topology. We can write $\mathbb{R}^\omega$ as the union of the set A consisting of all bounded sequences of real numbers, and the set B of all unbounded sequences. These sets are disjoint, and each is open in the box topology. For if $\mathbf{a}$ is a point of $\mathbb{R}^\omega$, the open set

$$U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$$

consists entirely of bounded sequences if $\mathbf{a}$ is bounded, and of unbounded sequences if $\mathbf{a}$ if unbounded. Thus, even though $\mathbb{R}$ is connected (as we shall prove in the next section), $\mathbb{R}^\omega$ is not connected in the box topology.

EXAMPLE 7. Now consider $\mathbb{R}^\omega$ in the product topology. Assuming that $\mathbb{R}$ is connected, we show that $\mathbb{R}^\omega$ is connected. Let $\tilde{\mathbb{R}}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ such that $x_i = 0$ for $i > n$. The space $\tilde{\mathbb{R}}^n$ is clearly homeomorphic to $\mathbb{R}^n$, so that it is connected, by the preceding theorem. It follows that the space $\mathbb{R}^\infty$ that is the union of the spaces $\tilde{\mathbb{R}}^n$ is connected, for these spaces have the point $\mathbf{0} = (0, 0, \ldots)$ in common. We show that the closure of $\mathbb{R}^\infty$ equals all of $\mathbb{R}^\omega$, from which it follows that $\mathbb{R}^\omega$ is connected as well.

Let $\mathbf{a} = (a_1, a_2, \ldots)$ be a point of $\mathbb{R}^\omega$. Let $U = \prod U_i$ be a basis element for the product topology that contains $\mathbf{a}$. We show that U intersects $\mathbb{R}^\infty$. There is an integer N such that $U_i = \mathbb{R}$ for $i > N$. Then the point

$$\mathbf{x} = (a_1, \ldots, a_n, 0, 0, \ldots)$$

of $\mathbb{R}^\infty$ belongs to U, since $a_i \in U_i$ for all i, and $0 \in U_i$ for $i > N$.

The argument just given generalizes to show that an arbitrary product of connected spaces is connected in the product topology. Since we shall not need this result, we leave the proof to the exercises.

Exercises

1. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}' \supset \mathcal{T}$, what does connectedness of X in one topology imply about connectedness in the other?

2. Let $\{A_n\}$ be a sequence of connected subspaces of X, such that $A_n \cap A_{n+1} \neq \varnothing$ for all n. Show that $\bigcup A_n$ is connected.

3. Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X. Show that if $A \cap A_\alpha \neq \varnothing$ for all α, then $A \cup (\bigcup A_\alpha)$ is connected.

4. Show that if X is an infinite set, it is connected in the finite complement topology.

5. A space is ***totally disconnected*** if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

6. Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects Bd A.

7. Is the space $\mathbb{R}_\ell$ connected? Justify your answer.

8. Determine whether or not $\mathbb{R}^\omega$ is connected in the uniform topology.

9. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that

$$(X \times Y) - (A \times B)$$

is connected.

10. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of connected spaces; let X be the product space

$$X = \prod_{\alpha \in J} X_\alpha.$$

Let $\mathbf{a} = (a_\alpha)$ be a fixed point of X.

(a) Given any finite subset K of J, let X_K denote the subspace of X consisting of all points $\mathbf{x} = (x_\alpha)$ such that $x_\alpha = a_\alpha$ for $\alpha \notin K$. Show that X_K is connected.

(b) Show that the union Y of the spaces X_K is connected.

(c) Show that X equals the closure of Y; conclude that X is connected.

11. Let $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected.

12. Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.

§24 Connected Subspaces of the Real Line

The theorems of the preceding section show us how to construct new connected spaces out of given ones. But where can we find some connected spaces to start with? The best place to begin is the real line. We shall prove that $\mathbb{R}$ is connected, and so are the intervals and rays in $\mathbb{R}$.

One application is the intermediate value theorem of calculus, suitably generalized. Another is the result that such familiar spaces as balls and spheres in euclidean space are connected; the proof involves a new notion, called *path connectedness*, which we also discuss.

The fact that intervals and rays in $\mathbb{R}$ are connected may be familiar to you from analysis. We prove it again here, in generalized form. It turns out that this fact does not depend on the algebraic properties of $\mathbb{R}$, but only on its order properties. To make this clear, we shall prove the theorem for an arbitrary ordered set that has the order properties of $\mathbb{R}$. Such a set is called a *linear continuum*.

Definition. A simply ordered set L having more than one element is called a ***linear continuum*** if the following hold:

(1) L has the least upper bound property.

(2) If $x < y$, there exists z such that $x < z < y$.

Theorem 24.1. *If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L.*

Proof. Recall that a subspace Y of L is said to be *convex* if for every pair of points a, b of Y with $a < b$, the entire interval $[a, b]$ of points of L lies in Y. We prove that if Y is a convex subspace of L, then Y is connected.

So suppose that Y is the union of the disjoint nonempty sets A and B, each of which is open in Y. Choose $a \in A$ and $b \in B$; suppose for convenience that $a < b$. The interval $[a, b]$ of points of L is contained in Y. Hence $[a, b]$ is the union of the disjoint sets

$$A_0 = A \cap [a, b] \quad \text{and} \quad B_0 = B \cap [a, b],$$

each of which is open in $[a, b]$ in the subspace topology, which is the same as the order topology. The sets A_0 and B_0 are nonempty because $a \in A_0$ and $b \in B_0$. Thus, A_0 and B_0 constitute a separation of $[a, b]$.

Let $c = \sup A_0$. We show that c belongs neither to A_0 nor to B_0, which contradicts the fact that $[a, b]$ is the union of A_0 and B_0.

Case 1. Suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$. In either case, it follows from the fact that B_0 is open in $[a, b]$ that there is some interval of the form $(d, c]$ contained in B_0. If $c = b$, we have a contradiction at once, for d is a smaller upper bound on A_0 than c. If $c < b$, we note that $(c, b]$ does not intersect A_0

(because c is an upper bound on A_0). Then

$$(d, b] = (d, c] \cup (c, b]$$

does not intersect A_0. Again, d is a smaller upper bound on A_0 than c, contrary to construction. See Figure 24.1.

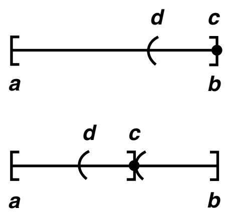

Figure 24.1

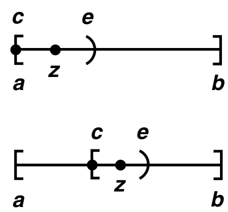

Figure 24.2

Case 2. Suppose that $c \in A_0$. Then $c \neq b$, so either $c = a$ or $a < c < b$. Because A_0 is open in $[a, b]$, there must be some interval of the form $[c, e)$ contained in A_0. See Figure 24.2. Because of order property (2) of the linear continuum L, we can choose a point z of L such that $c < z < e$. Then $z \in A_0$, contrary to the fact that c is an upper bound for A_0. ∎

Corollary 24.2. *The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.*

As an application, we prove the intermediate value theorem of calculus, suitably generalized.

Theorem 24.3 (Intermediate value theorem). *Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.*

The intermediate value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

Proof. Assume the hypotheses of the theorem. The sets

$$A = f(X) \cap (-\infty, r) \quad \text{and} \quad B = f(X) \cap (r, +\infty)$$

are disjoint, and they are nonempty because one contains $f(a)$ and the other contains $f(b)$. Each is open in $f(X)$, being the intersection of an open ray in Y with $f(X)$. If there were no point c of X such that $f(c) = r$, then $f(X)$ would be the union of the sets A and B. Then A and B would constitute a separation of $f(X)$, contradicting the fact that the image of a connected space under a continuous map is connected. ∎

EXAMPLE 1. One example of a linear continuum different from $\mathbb{R}$ is the ordered square. We check the least upper bound property. (The second property of a linear continuum is trivial to check.) Let A be a subset of $I \times I$; let $\pi_1 : I \times I \to I$ be projection on the first coordinate; let $b = \sup \pi_1(A)$. If $b \in \pi_1(A)$, then A intersects the subset $b \times I$ of $I \times I$. Because $b \times I$ has the order type of I, the set $A \cap (b \times I)$ will have a least upper bound $b \times c$, which will be the least upper bound of A. See Figure 24.3. If $b \notin \pi_1(A)$, then $b \times 0$ is the least upper bound of A; no element of the form $b' \times c$ with $b' < b$ can be an upper bound for A, for then b' would be an upper bound for $\pi_1(A)$.

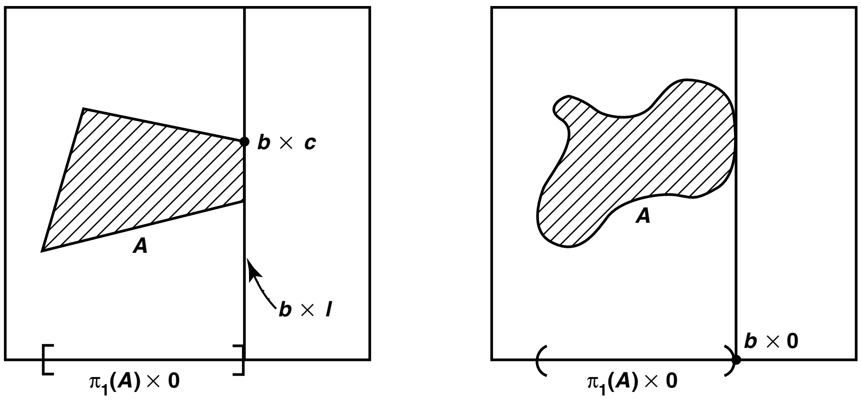

Figure 24.3

EXAMPLE 2. If X is a well-ordered set, then $X \times [0, 1)$ is a linear continuum in the dictionary order; this we leave to you to check. This set can be thought of as having been constructed by "fitting in" a set of the order type of $(0, 1)$ immediately following each element of X.

Connectedness of intervals in $\mathbb{R}$ gives rise to an especially useful criterion for showing that a space X is connected; namely, the condition that every pair of points of X can be joined by a *path* in X:

Definition. Given points x and y of the space X, a ***path*** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be ***path connected*** if every pair of points of X can be joined by a path in X.

It is easy to see that a path-connected space X is connected. Suppose $X = A \cup B$ is a separation of X. Let $f : [a, b] \to X$ be any path in X. Being the continuous image of a connected set, the set $f([a, b])$ is connected, so that it lies entirely in either A or B. Therefore, there is no path in X joining a point of A to a point of B, contrary to the assumption that X is path connected.

The converse does not hold; a connected space need not be path connected. See Examples 6 and 7 following.