PEARSON NEW INTERNATIONAL EDITION

Fundamentals of Differential Equations and Boundary Value Problems

Nagle Saff Snider

Sixth Edition

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so the rate of change of the mass of salt in tank A is

$$\frac{dx}{dt} = \frac{2}{24}y - \frac{8}{24}x = \frac{1}{12}y - \frac{1}{3}x \ .$$

The rate of change of salt in tank B is determined by the same interconnecting pipes *and* by the drain pipe, carrying away $6y/24$ kg/min:

$$\frac{dy}{dt} = \frac{8}{24}x - \frac{2}{24}y - \frac{6}{24}y = \frac{1}{3}x - \frac{1}{3}y \ .$$

The interconnected tanks are thus governed by a *system* of differential equations:

(1)
$$x' = -\frac{1}{3}x + \frac{1}{12}y \ ,$$
$$y' = \frac{1}{3}x - \frac{1}{3}y \ .$$

Although both unknowns $x(t)$ and $y(t)$ appear in each of equations (1) (they are "coupled"), the structure is so transparent that we can obtain an equation for y alone by solving the second equation for x,

(2) $$x = 3y' + y \ ,$$

and substituting (2) in the first equation to eliminate x:

$$(3y' + y)' = -\frac{1}{3}(3y' + y) + \frac{1}{12}y \ ,$$
$$3y'' + y' = -y' - \frac{1}{3}y + \frac{1}{12}y \ ,$$

or

$$3y'' + 2y' + \frac{1}{4}y = 0 \ .$$

Since the auxiliary equation

$$3r^2 + 2r + \frac{1}{4} = 0$$

has roots $-1/2$, $-1/6$, a general solution is given by

(3) $$y(t) = c_1e^{-t/2} + c_2e^{-t/6} \ .$$

Having determined y, we use equation (2) to deduce a formula for x:

(4) $$x(t) = 3\left(-\frac{c_1}{2}e^{-t/2} - \frac{c_2}{6}e^{-t/6}\right) + c_1e^{-t/2} + c_2e^{-t/6} = -\frac{1}{2}c_1e^{-t/2} + \frac{1}{2}c_2e^{-t/6} \ .$$

Formulas (3) and (4) contain two undetermined parameters, c_1 and c_2, which can be adjusted to meet the specified initial conditions:

$$x(0) = -\frac{1}{2}c_1 + \frac{1}{2}c_2 = x_0 \ , \qquad y(0) = c_1 + c_2 = y_0 \ ,$$

or

$$c_1 = \frac{y_0 - 2x_0}{2}\,, \qquad c_2 = \frac{y_0 + 2x_0}{2}\,.$$

Thus, the mass of salt in tanks A and B at time t are, respectively,

$$\begin{aligned} x(t) &= -\left(\frac{y_0 - 2x_0}{4}\right)e^{-t/2} + \left(\frac{y_0 + 2x_0}{4}\right)e^{-t/6}\,, \\ y(t) &= \left(\frac{y_0 - 2x_0}{2}\right)e^{-t/2} + \left(\frac{y_0 + 2x_0}{2}\right)e^{-t/6}\,. \end{aligned} \tag{5}$$

The ad hoc elimination procedure that we used to solve this example will be generalized and formalized in the next section, to find solutions of all *linear systems with constant coefficients.* Furthermore, in later sections we will show how to extend our numerical algorithms for first-order equations to *general* systems and will consider applications to coupled oscillators and electrical systems.

It is interesting to note from (5) that all solutions of the interconnected-tanks problem tend to the constant solution $x(t) \equiv 0$, $y(t) \equiv 0$ as $t \to +\infty$. (This is of course consistent with our physical expectations.) This constant solution will be identified as a *stable equilibrium solution* in Section 4, in which we introduce phase plane analysis. It turns out that, for a general class of systems, equilibria can be identified and classified so as to give qualitative information about the other solutions even when we cannot solve the system explicitly.

2 DIFFERENTIAL OPERATORS AND THE ELIMINATION METHOD FOR SYSTEMS

The notation $y'(t) = \dfrac{dy}{dt} = \dfrac{d}{dt}y$ was devised to suggest that the derivative of a function y is the result of *operating* on the function y with the differentiation operator $\dfrac{d}{dt}$. Indeed, second derivatives are formed by iterating the operation: $y''(t) = \dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\dfrac{d}{dt}y$. Commonly, the symbol D is used instead of $\dfrac{d}{dt}$, and the second-order differential equation

$$y'' + 4y' + 3y = 0$$

is represented† by

$$D^2y + 4Dy + 3y = (D^2 + 4D + 3)[y] = 0\,.$$

So, we have implicitly adopted the convention that the operator "product," D times D, is interpreted as the *composition* of D with itself, when it operates on functions: D^2y means $D(D[y])$; i.e., the second derivative. Similarly, the product $(D + 3)(D + 1)$ operates on a function via

$$\begin{aligned} (D+3)(D+1)[y] &= (D+3)\big[(D+1)[y]\big] = (D+3)[y' + y] \\ &= D[y' + y] + 3[y' + y] \\ &= (y'' + y') + (3y' + 3y) = y'' + 4y' + 3y = (D^2 + 4D + 3)[y]\,. \end{aligned}$$

†Some authors utilize the identity operator I, defined by $I[y] = y$, and write more formally $D^2 + 4D + 3I$ instead of $D^2 + 4D + 3$.

Thus, $(D+3)(D+1)$ is the same operator as D^2+4D+3; when they are applied to twice-differentiable functions, the results are identical.

Example 1 Show that the operator $(D+1)(D+3)$ is also the same as D^2+4D+3.

Solution For any twice-differentiable function $y(t)$, we have

$$\begin{aligned}(D+1)(D+3)[y] &= (D+1)\big[(D+3)[y]\big] = (D+1)[y'+3y] \\ &= D[y'+3y] + 1[y'+3y] = (y''+3y') + (y'+3y) \\ &= y''+4y'+3y = (D^2+4D+3)[y]\ .\end{aligned}$$

Hence, $(D+1)(D+3) = D^2+4D+3$. ◆

Since $(D+1)(D+3) = (D+3)(D+1) = D^2+4D+3$, it is tempting to generalize and propose that one can treat expressions like aD^2+bD+c as if they were ordinary polynomials in D. This is true, as long as we restrict the coefficients a, b, c to be *constants*. The following example, which has *variable* coefficients, is instructive.

Example 2 Show that $(D+3t)D$ is *not* the same as $D(D+3t)$.

Solution With $y(t)$ as before,

$$(D+3t)D[y] = (D+3t)[y'] = y''+3ty'\ ;$$

$$D(D+3t)[y] = D[y'+3ty] = y''+3y+3ty'\ .$$

They are not the same! ◆

Because the coefficient $3t$ is not a constant, it "interrupts" the interaction of the differentiation operator D with the function $y(t)$. As long as we only deal with expressions like aD^2+bD+c with *constant* coefficients $a, b,$ and $c,$ the "algebra" of differential operators follows the same rules as the algebra of polynomials. (See Problem 39 for elaboration on this point.)

This means that the familiar elimination method, used for solving *algebraic* systems like

$$\begin{aligned}3x - 2y + z &= 4\ ,\\ x + y - z &= 0\ ,\\ 2x - y + 3z &= 6\ ,\end{aligned}$$

can be adapted to solve any system of *linear differential equations with constant coefficients.* In fact, we used this approach in solving the system that arose in the interconnected tanks problem of Section 1. Our goal in this section is to formalize this **elimination method** so that we can tackle more general linear constant coefficient systems.

We first demonstrate how the method applies to a linear system of two first-order differential equations of the form

$$a_1x'(t) + a_2x(t) + a_3y'(t) + a_4y(t) = f_1(t)\ ,$$

$$a_5x'(t) + a_6x(t) + a_7y'(t) + a_8y(t) = f_2(t)\ ,$$

where $a_1, a_2, \ldots, a_8$ are constants and $x(t), y(t)$ is the function pair to be determined. In operator notation this becomes

$$(a_1D + a_2)[x] + (a_3D + a_4)[y] = f_1\ ,$$

$$(a_5D + a_6)[x] + (a_7D + a_8)[y] = f_2\ .$$

Example 3 Solve the system

$$(1) \qquad \begin{aligned} x'(t) &= 3x(t) - 4y(t) + 1 \ , \\ y'(t) &= 4x(t) - 7y(t) + 10t \ . \end{aligned}$$

Solution The alert reader may observe that since y' is absent from the first equation, we could use the latter to express y in terms of x and x' and substitute into the second equation to derive an "uncoupled" equation containing only x and its derivatives. However, this simple trick will not work on more general systems (Problem 18 is an example).

To utilize the elimination method, we first write the system using the operator notation:

$$(2) \qquad \begin{aligned} (D-3)[x] + 4y &= 1 \ , \\ -4x + (D+7)[y] &= 10t \ . \end{aligned}$$

Imitating the elimination procedure for algebraic systems, we can eliminate x from this system by adding 4 times the first equation to $(D-3)$ applied to the second equation. This gives

$$\big(16 + (D-3)(D+7)\big)[y] = 4 \cdot 1 + (D-3)[10t] = 4 + 10 - 30t \ ,$$

which simplifies to

$$(3) \qquad (D^2 + 4D - 5)[y] = 14 - 30t \ .$$

Now equation (3) is just a second-order linear equation in y with constant coefficients that has the general solution

$$(4) \qquad y(t) = C_1 e^{-5t} + C_2 e^{t} + 6t + 2 \ ,$$

which can be found using undetermined coefficients.

To find $x(t)$, we have two options.

Method 1. We return to system (2) and eliminate y. This is accomplished by "multiplying" the first equation in (2) by $(D+7)$ and the second equation by -4 and then adding to obtain

$$(D^2 + 4D - 5)[x] = 7 - 40t \ .$$

This equation can likewise be solved using undetermined coefficients to yield

$$(5) \qquad x(t) = K_1 e^{-5t} + K_2 e^{t} + 8t + 5 \ ,$$

where we have taken K_1 and K_2 to be the arbitrary constants, which are not necessarily the same as C_1 and C_2 used in formula (4).

It is reasonable to expect that system (1) will involve only *two* arbitrary constants, since it consists of two first-order equations. Thus, the four constants C_1, C_2, K_1, and K_2 are not independent. To determine the relationships, we substitute the expressions for $x(t)$ and $y(t)$ given in (4) and (5) into one of the equations in (1), say, the first one. This yields

$$\begin{aligned} &-5K_1 e^{-5t} + K_2 e^{t} + 8 = \\ &\qquad 3K_1 e^{-5t} + 3K_2 e^{t} + 24t + 15 - 4C_1 e^{-5t} - 4C_2 e^{t} - 24t - 8 + 1 \ , \end{aligned}$$

which simplifies to

$$(4C_1 - 8K_1)e^{-5t} + (4C_2 - 2K_2)e^{t} = 0 \ .$$

Because e^t and e^{-5t} are linearly independent functions on any interval, this last equation holds for all t only if

$$4C_1 - 8K_1 = 0 \quad \text{and} \quad 4C_2 - 2K_2 = 0 \ .$$

Therefore, $K_1 = C_1/2$ and $K_2 = 2C_2$.

A solution to system (1) is then given by the pair

$$\text{(6)} \qquad x(t) = \frac{1}{2}C_1e^{-5t} + 2C_2e^t + 8t + 5 \ , \qquad y(t) = C_1e^{-5t} + C_2e^t + 6t + 2 \ .$$

As you might expect, this pair is a **general solution** to (1) in the sense that *any* solution to (1) can be expressed in this fashion.

Method 2. A simpler method for determining $x(t)$ once $y(t)$ is known is to use the system to obtain an equation for $x(t)$ in terms of $y(t)$ and $y'(t)$. In this example we can directly solve the second equation in (1) for $x(t)$:

$$x(t) = \frac{1}{4}y'(t) + \frac{7}{4}y(t) - \frac{5}{2}t \ .$$

Substituting $y(t)$ as given in (4) yields

$$\begin{aligned} x(t) &= \frac{1}{4}\left[-5C_1e^{-5t} + C_2e^t + 6\right] + \frac{7}{4}\left[C_1e^{-5t} + C_2e^t + 6t + 2\right] - \frac{5}{2}t \\ &= \frac{1}{2}C_1e^{-5t} + 2C_2e^t + 8t + 5 \ , \end{aligned}$$

which agrees with (6). ◆

The above procedure works, more generally, for any linear system of two equations and two unknowns with *constant coefficients* regardless of the order of the equations. For example, if we let L_1, L_2, L_3, and L_4 denote linear differential operators with constant coefficients (i.e., polynomials in D), then the method can be applied to the linear system

$$\begin{aligned} L_1[x] + L_2[y] &= f_1 \ , \\ L_3[x] + L_4[y] &= f_2 \ . \end{aligned}$$

Because the system has constant coefficients, the operators commute (e.g., $L_2L_4 = L_4L_2$) and we can eliminate variables in the usual algebraic fashion. Eliminating the variable y gives

$$\text{(7)} \qquad (L_1L_4 - L_2L_3)[x] = g_1 \ ,$$

where $g_1 := L_4[f_1] - L_2[f_2]$. Similarly, eliminating the variable x yields

$$\text{(8)} \qquad (L_1L_4 - L_2L_3)[y] = g_2 \ ,$$

where $g_2 := L_1[f_2] - L_3[f_1]$. Now if $L_1L_4 - L_2L_3$ is a differential operator of order n, then a general solution for (7) contains n arbitrary constants, and a general solution for (8) also contains n arbitrary constants. Thus, a total of $2n$ constants arise. However, as we saw in Example 3, there are only n of these that are independent for the system; the remaining constants can be expressed in terms of these.† The pair of general solutions to (7) and (8) written in terms of the n independent constants is called a **general solution for the system.**

†For a proof of this fact, see *Ordinary Differential Equations*, by M. Tenenbaum and H. Pollard (Dover, New York, 1985). Chapter 7.

If it turns out that $L_1L_4 - L_2L_3$ is the zero operator, the system is said to be **degenerate.** As with the anomalous problem of solving for the points of intersection of two parallel or coincident lines, a degenerate system may have no solutions, or if it does possess solutions, they may involve any number of arbitrary constants (see Problems 23 and 24).

Elimination Procedure for 2 × 2 Systems

To find a general solution for the system

$$L_1[x] + L_2[y] = f_1 \ ,$$
$$L_3[x] + L_4[y] = f_2 \ ,$$

where L_1, L_2, L_3, and L_4 are polynomials in $D = d/dt$:

(a) Make sure that the system is written in operator form.
(b) Eliminate one of the variables, say, y, and solve the resulting equation for $x(t)$. If the system is degenerate, stop! A separate analysis is required to determine whether or not there are solutions.
(c) (*Shortcut*) If possible, use the system to derive an equation that involves $y(t)$ but not its derivatives. [Otherwise, go to step (d).] Substitute the found expression for $x(t)$ into this equation to get a formula for $y(t)$. The expressions for $x(t)$, $y(t)$ give the desired general solution.
(d) Eliminate x from the system and solve for $y(t)$. [Solving for $y(t)$ gives more constants—in fact, twice as many as needed.]
(e) Remove the extra constants by substituting the expressions for $x(t)$ and $y(t)$ into one or both of the equations in the system. Write the expressions for $x(t)$ and $y(t)$ in terms of the remaining constants.

Example 4 Find a general solution for

$$x''(t) + y'(t) - x(t) + y(t) = -1 \ ,$$
$$x'(t) + y'(t) - x(t) = t^2 \ . \qquad (9)$$

Solution We begin by expressing the system in operator notation:

$$(D^2 - 1)[x] + (D + 1)[y] = -1 \ ,$$
$$(D - 1)[x] + D[y] = t^2 \ . \qquad (10)$$

Here $L_1 \coloneqq D^2 - 1$, $L_2 \coloneqq D + 1$, $L_3 \coloneqq D - 1$, and $L_4 \coloneqq D$.

Eliminating y gives [see (7)]:

$$\big((D^2 - 1)D - (D + 1)(D - 1)\big)[x] = D[-1] - (D + 1)[t^2] \ ,$$

which reduces to

$$(D^2 - 1)(D - 1)[x] = -2t - t^2 \ ,$$
$$(D - 1)^2(D + 1)[x] = -2t - t^2 \ . \qquad (11)$$

Since $(D - 1)^2(D + 1)$ is third order, we should expect three arbitrary constants in a general solution to system (9).