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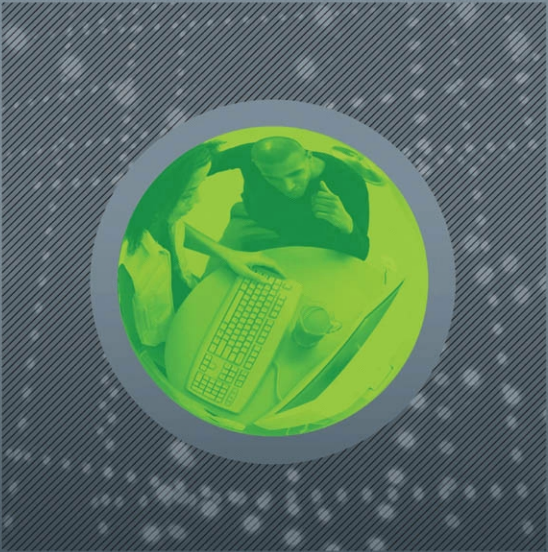

Discrete and Combinatorial Mathematics
An Applied Introduction
Ralph Grimaldi
Fifth Edition

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the left-most endpoint, three branches originate — one for each of the elements of A. Then from each point, labeled 2, 3, 4, two branches emanate — one for each of the elements 4, 5 of B. The six ordered pairs at the right endpoints constitute the elements (ordered pairs) of $A \times B$. Part (b) of the figure provides a tree diagram to demonstrate the construction of $B \times A$. Finally, the tree diagram in Fig. 1 (c) shows us how to envision the construction of $A \times B \times C$, and demonstrates that $|A \times B \times C| = 12 = 3 \times 2 \times 2 = |A||B||C|$.

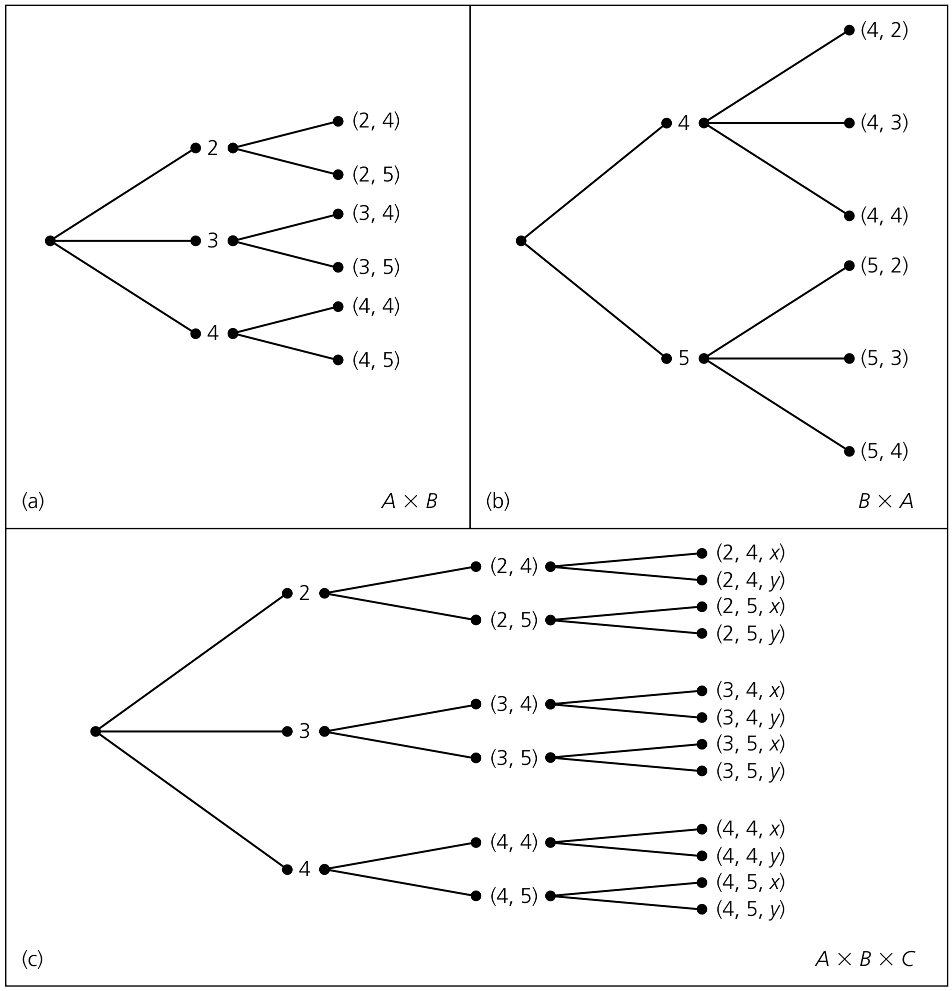

Figure 1

In addition to their tie-in with Cartesian products, tree diagrams also arise in other situations.

EXAMPLE 4

At the Wimbledon Tennis Championships, women play at most three sets in a match. The winner is the first to win two sets. If we let N and E denote the two players, the tree diagram in Fig. 2 indicates the six ways in which this match can be won. For example, the starred line segment (edge) indicates that player E won the first set. The double-starred edge indicates that player N has won the match by winning the first and third sets.

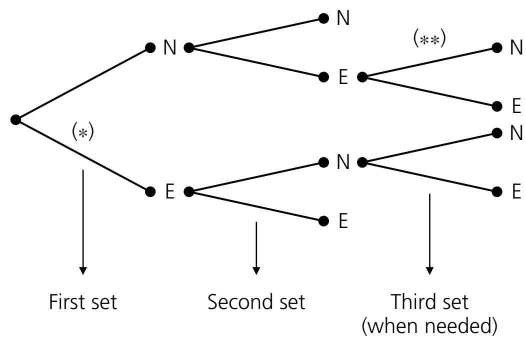

Figure 2

Tree diagrams are examples of a general structure called a *tree*. Trees and graphs are important structures that arise in computer science and optimization theory. These will be investigated in later chapters.

For the cross product of two sets, we find the subsets of this structure of great interest.

Definition 2

For sets A, B, any subset of $A \times B$ is called a (*binary*) *relation* from A to B. Any subset of $A \times A$ is called a (*binary*) *relation* on A.

Since we will primarily deal with binary relations, for us the word "relation" will mean binary relation, unless something otherwise is specified.

EXAMPLE 5

With A, B as in Example 1, the following are some of the relations from A to B.

a) $\emptyset$
b) $\{(2, 4)\}$
c) $\{(2, 4), (2, 5)\}$
d) $\{(2, 4), (3, 4), (4, 4)\}$
e) $\{(2, 4), (3, 4), (4, 5)\}$
f) $A \times B$

Since $|A \times B| = 6$, it follows from Definition 2 that there are 2^6 possible relations from A to B (for there are 2^6 possible subsets of $A \times B$).

For finite sets A, B with $|A| = m$ and $|B| = n$, there are 2^{mn} relations from A to B, including the empty relation as well as the relation $A \times B$ itself.

There are also $2^{nm} (= 2^{mn})$ relations from B to A, one of which is also $\emptyset$ and another of which is $B \times A$. The reason we get the same number of relations from B to A as we have from A to B is that any relation $\mathcal{R}_1$ from B to A can be obtained from a unique relation $\mathcal{R}_2$ from A to B by simply reversing the components of each ordered pair in $\mathcal{R}_2$ (and vice versa).

EXAMPLE 6

For $B = \{1, 2\}$, let $A = \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. The following is an example of a *relation* on A: $\mathcal{R} = \{(\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{1\}), (\{1\}, \{1, 2\}), (\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\})\}$. We can say that the relation $\mathcal{R}$ is the *subset relation* where $(C, D) \in \mathcal{R}$ if and only if $C, D \subseteq B$ and $C \subseteq D$.

EXAMPLE 7

With $A = \mathbf{Z}^+$, we may define a relation $\mathcal{R}$ on set A as $\{(x, y) | x \leq y\}$. This is the familiar "is less than or equal to" relation for the set of positive integers. It can be represented graphically as the set of points, with positive integer components, located on or above the line $y = x$ in the Euclidean plane, as partially shown in Fig. 3. Here we cannot list the entire relation as we did in Example 6, but we note, for example, that $(7, 7), (7, 11) \in \mathcal{R}$, but $(8, 2) \notin \mathcal{R}$. The fact that $(7, 11) \in \mathcal{R}$ can also be denoted by $7 \mathcal{R} 11$; $(8, 2) \notin \mathcal{R}$ becomes $8 \not\mathcal{R} 2$. Here $7 \mathcal{R} 11$ and $8 \not\mathcal{R} 2$ are examples of the *infix* notation for a relation.

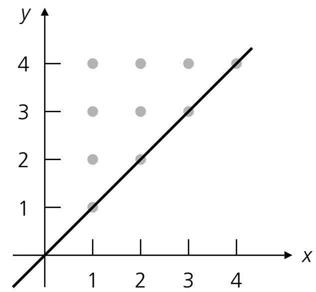

Figure 3

Our last example helps us to review the idea of a recursively defined set.

EXAMPLE 8

Let $\mathcal{R}$ be the subset of $\mathbf{N} \times \mathbf{N}$ where $\mathcal{R} = \{(m, n) | n = 7m\}$. Consequently, among the ordered pairs in $\mathcal{R}$ one finds $(0, 0)$, $(1, 7)$, $(11, 77)$, and $(15, 105)$. This relation $\mathcal{R}$ on $\mathbf{N}$ can also be given recursively by

1) $(0, 0) \in \mathcal{R}$; and

2) If $(s, t) \in \mathcal{R}$, then $(s + 1, t + 7) \in \mathcal{R}$.

We use the recursive definition to show that the ordered pair $(3, 21)$ (from $\mathbf{N} \times \mathbf{N}$) is in $\mathcal{R}$. Our derivation is as follows: From part (1) of the recursive definition we start with $(0, 0) \in \mathcal{R}$. Then part (2) of the definition gives us

i) $(0, 0) \in \mathcal{R} \Rightarrow (0 + 1, 0 + 7) = (1, 7) \in \mathcal{R}$;

ii) $(1, 7) \in \mathcal{R} \Rightarrow (1 + 1, 7 + 7) = (2, 14) \in \mathcal{R}$; and

iii) $(2, 14) \in \mathcal{R} \Rightarrow (2 + 1, 14 + 7) = (3, 21) \in \mathcal{R}$.

We close this section with these final observations.

1) For any set A, $A \times \emptyset = \emptyset$. (If $A \times \emptyset \neq \emptyset$, let $(a, b) \in A \times \emptyset$. Then $a \in A$ and $b \in \emptyset$. Impossible!) Likewise, $\emptyset \times A = \emptyset$.

2) The Cartesian product and the binary operations of union and intersection are interrelated in the following theorem.

THEOREM 1

For any sets $A, B, C \subseteq \mathcal{U}$:

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof: We prove part (a) and leave the other parts for the reader. We use the same concept of set equality even though the elements here are ordered pairs. For all $a, b \in \mathcal{U}$, $(a, b) \in A \times (B \cap C) \Longleftrightarrow a \in A$ and $b \in B \cap C \Longleftrightarrow a \in A$ and $b \in B, C \Longleftrightarrow a \in A, b \in B$ and $a \in A, b \in C \Longleftrightarrow (a, b) \in A \times B$ and $(a, b) \in A \times C \Longleftrightarrow (a, b) \in (A \times B) \cap (A \times C)$.

EXERCISES 1

1. If $A = \{1, 2, 3, 4\}$, $B = \{2, 5\}$, and $C = \{3, 4, 7\}$, determine $A \times B$; $B \times A$; $A \cup (B \times C)$; $(A \cup B) \times C$; $(A \times C) \cup (B \times C)$.

2. If $A = \{1, 2, 3\}$, and $B = \{2, 4, 5\}$, give examples of (a) three nonempty relations from A to B; (b) three nonempty relations on A.

3. For A, B as in Exercise 2, determine the following: (a) $|A \times B|$; (b) the number of relations from A to B; (c) the number of relations on A; (d) the number of relations from A to B that contain (1, 2) and (1, 5); (e) the number of relations from A to B that contain exactly five ordered pairs; and (f) the number of relations on A that contain at least seven elements.

4. For which sets A, B is it true that $A \times B = B \times A$?

5. Let A, B, C, D be nonempty sets.

a) Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$.

b) What happens to the result in part (a) if any of the sets A, B, C, D is empty?

6. The men's final at Wimbledon is won by the first player to win three sets of the five-set match. Let C and M denote the players. Draw a tree diagram to show all the ways in which the match can be decided.

7. a) If $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$, how many elements are there in $\mathcal{P}(A \times B)$?

b) Generalize the result in part (a).

8. Logic chips are taken from a container, tested individually, and labeled defective or good. The testing process is continued until either two defective chips are found or five chips are tested in total. Using a tree diagram, exhibit a sample space for this process.

9. Complete the proof of Theorem 1.

10. A rumor is spread as follows. The originator calls two people. Each of these people phones three friends, each of whom in turn calls five associates. If no one receives more than one call, and no one calls the originator, how many people now know the rumor? How many phone calls were made?

11. For $A, B, C \subseteq \mathcal{U}$, prove that

$$A \times (B - C) = (A \times B) - (A \times C).$$

12. Let A, B be sets with $|B| = 3$. If there are 4096 relations from A to B, what is $|A|$?

13. Let $\mathcal{R} \subseteq \mathbf{N} \times \mathbf{N}$ where $(m, n) \in \mathcal{R}$ if (and only if) $n = 5m + 2$. (a) Give a recursive definition for $\mathcal{R}$. (b) Use the recursive definition from part (a) to show that $(4, 22) \in \mathcal{R}$.

14. a) Give a recursive definition for the relation $\mathcal{R} \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+$ where $(m, n) \in \mathcal{R}$ if (and only if) $m \geq n$.

b) From the definition in part (a) verify that (5, 2) and (4, 4) are in $\mathcal{R}$.

2 Functions: Plain and One-to-One

In this section we concentrate on a special kind of relation called a *function*. One finds functions in many different settings throughout mathematics and computer science.

Definition 3 For nonempty sets A, B, a *function*, or *mapping*, *f from A to B*, denoted $f\colon A \to B$, is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

We often write $f(a) = b$ when (a, b) is an ordered pair in the function f. For $(a, b) \in f$, b is called *the image* of a under f, whereas a is a *preimage* of b. In addition, the definition suggests that f is a method for *associating* with each $a \in A$ the *unique* element $f(a) = b \in B$. Consequently, $(a, b), (a, c) \in f$ implies $b = c$.

EXAMPLE 9

For $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$, $f = \{(1, w), (2, x), (3, x)\}$ is a function, and consequently a relation, from A to B. $\mathcal{R}_1 = \{(1, w), (2, x)\}$ and $\mathcal{R}_2 = \{(1, w), (2, w), (2, x), (3, z)\}$ are relations, but not functions, from A to B. (Why?)

Definition 4

For the function $f: A \to B$, A is called the *domain* of f and B the *codomain* of f. The subset of B consisting of those elements that appear as second components in the ordered pairs of f is called the *range* of f and is also denoted by $f(A)$ because it is the set of images (of the elements of A) under f.

In Example 9, the domain of $f = \{1, 2, 3\}$, the codomain of $f = \{w, x, y, z\}$, and the range of $f = f(A) = \{w, x\}$.

A pictorial representation of these ideas appears in Fig. 4. This diagram suggests that a may be regarded as an *input* that is *transformed* by f into the corresponding *output*, $f(a)$. In this context, a C++ compiler can be thought of as a function that transforms a source program (the input) into its corresponding object program (the output).

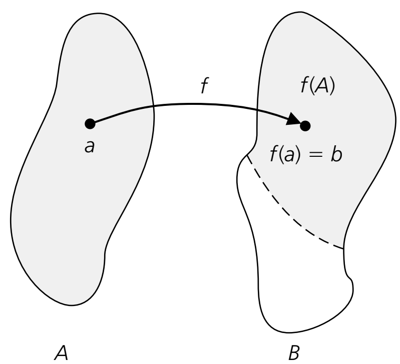

Figure 4

EXAMPLE 10

Many interesting functions arise in computer science.

a) A common function encountered is the *greatest integer function,* or *floor function.* This function $f: \mathbf{R} \to \mathbf{Z}$, is given by

$$f(x) = \lfloor x \rfloor = \text{the greatest integer less than or equal to } x.$$

Consequently, $f(x) = x$, if $x \in \mathbf{Z}$; and, when $x \in \mathbf{R} - \mathbf{Z}$, $f(x)$ is the integer to the immediate left of x on the real number line.

For this function we find that

1) $\lfloor 3.8 \rfloor = 3$, $\lfloor 3 \rfloor = 3$, $\lfloor -3.8 \rfloor = -4$, $\lfloor -3 \rfloor = -3$;

2) $\lfloor 7.1 + 8.2 \rfloor = \lfloor 15.3 \rfloor = 15 = 7 + 8 = \lfloor 7.1 \rfloor + \lfloor 8.2 \rfloor$; and

3) $\lfloor 7.7 + 8.4 \rfloor = \lfloor 16.1 \rfloor = 16 \neq 15 = 7 + 8 = \lfloor 7.7 \rfloor + \lfloor 8.4 \rfloor$.

b) A second function — one related to the floor function in part (a) — is the *ceiling function*. This function $g: \mathbf{R} \to \mathbf{Z}$ is defined by

$$g(x) = \lceil x \rceil = \text{the least integer greater than or equal to } x.$$

So $g(x) = x$ when $x \in \mathbf{Z}$, but when $x \in \mathbf{R} - \mathbf{Z}$, then $g(x)$ is the integer to the immediate right of x on the real number line. In dealing with the ceiling function one finds that

1) $\lceil 3 \rceil = 3,\ \lceil 3.01 \rceil = \lceil 3.7 \rceil = 4 = \lceil 4 \rceil,\ \lceil -3 \rceil = -3,\ \lceil -3.01 \rceil = \lceil -3.7 \rceil = -3;$

2) $\lceil 3.6 + 4.5 \rceil = \lceil 8.1 \rceil = 9 = 4 + 5 = \lceil 3.6 \rceil + \lceil 4.5 \rceil$; and

3) $\lceil 3.3 + 4.2 \rceil = \lceil 7.5 \rceil = 8 \neq 9 = 4 + 5 = \lceil 3.3 \rceil + \lceil 4.2 \rceil$.

c) The function trunc (for truncation) is another integer-valued function defined on $\mathbf{R}$. This function deletes the fractional part of a real number. For example, trunc(3.78) $= 3$, trunc(5) $= 5$, trunc(-7.22) $= -7$. Note that trunc(3.78) $= \lfloor 3.78 \rfloor = 3$ while trunc(-3.78) $= \lceil -3.78 \rceil = -3$.

d) In storing a matrix in a one-dimensional array, many computer languages use the *row major* implementation. Here, if $A = (a_{ij})_{m \times n}$ is an $m \times n$ matrix, the first row of A is stored in locations $1, 2, 3, \ldots, n$ of the array if we start with a_{11} in location 1. The entry a_{21} is then found in position $n + 1$, while entry a_{34} occupies position $2n + 4$ in the array. In order to determine the location of an entry a_{ij} from A, where $1 \leq i \leq m$, $1 \leq j \leq n$, one defines the *access function* f from the entries of A to the positions $1, 2, 3, \ldots, mn$ of the array. A formula for the access function here is $f(a_{ij}) = (i - 1)n + j$.

a_{11}	a_{12}	$\cdots$	a_{1n}	a_{21}	a_{22}	$\cdots$	a_{2n}	a_{31}	$\cdots$	a_{ij}	$\cdots$	a_{mn}
1	2	$\cdots$	n	$n+1$	$n+2$	$\cdots$	$2n$	$2n+1$	$\cdots$	$(i-1)n + j$	$\cdots$	$(m-1)n + n\ (= mn)$

EXAMPLE 11

We may use the floor and ceiling functions in parts (a) and (b), respectively, of Example 10 to restate some of the ideas we examined earlier.

a) When studying the division algorithm, we learned that for all $a, b \in \mathbf{Z}$, where $b > 0$, it was possible to find unique $q, r \in \mathbf{Z}$ with $a = qb + r$ and $0 \leq r < b$. Now we may add that $q = \left\lfloor \frac{a}{b} \right\rfloor$ and $r = a - \left\lfloor \frac{a}{b} \right\rfloor b$.

b) In Example 44 we found that the positive integer

$$29{,}338{,}848{,}000 = 2^8 3^5 5^3 7^3 11$$

has

$$60 = (5)(3)(2)(2)(1) = \left\lceil \frac{(8+1)}{2} \right\rceil \left\lceil \frac{(5+1)}{2} \right\rceil \left\lceil \frac{(3+1)}{2} \right\rceil \left\lceil \frac{(3+1)}{2} \right\rceil \left\lceil \frac{(1+1)}{2} \right\rceil$$

positive divisors that are perfect squares. In general, if $n \in \mathbf{Z}^+$ with $n > 1$, we know that we can write

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

where $k \in \mathbf{Z}^+$, p_i is prime for all $1 \leq i \leq k$, $p_i \neq p_j$ for all $1 \leq i < j \leq k$, and $e_i \in \mathbf{Z}^+$ for all $1 \leq i \leq k$. This is due to the Fundamental Theorem of Arithmetic. Then if $r \in \mathbf{Z}^+$, we find that the number of positive divisors of n that are perfect rth powers is $\prod_{i=1}^{k} \left\lceil \frac{e_i + 1}{r} \right\rceil$. When $r = 1$ we get $\prod_{i=1}^{k} \lceil e_i + 1 \rceil = \prod_{i=1}^{k} (e_i + 1)$, which is the number of positive divisors of n.