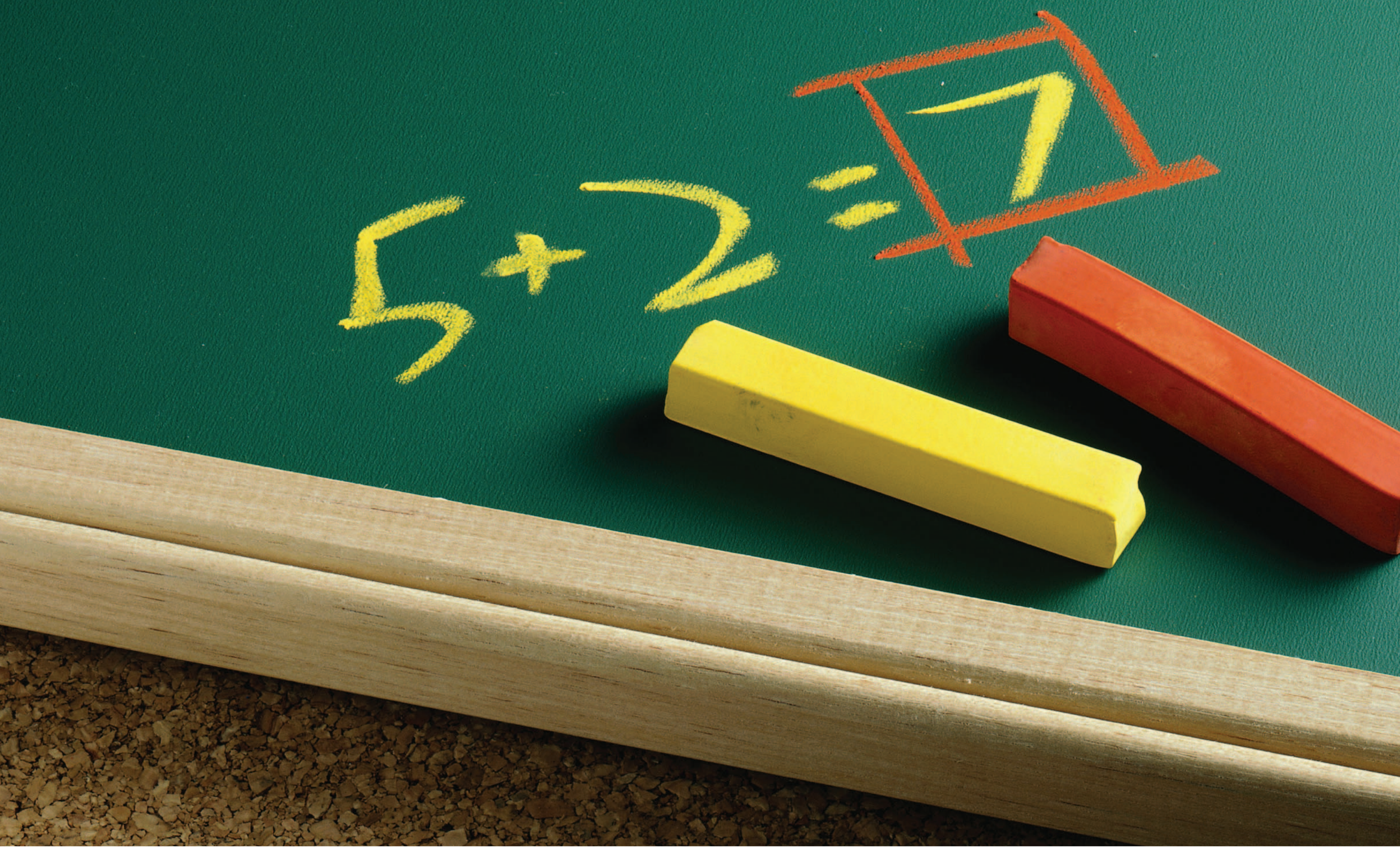
Thinking Mathematically

Robert Blitzer

Fifth Edition



Pearson New International Edition

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PEARSON

OBJECTIVES

1 | Define the integers.
2 | Graph integers on a number line.
3 | Use the symbols $<$ and $>$.
4 | Find the absolute value of an integer.
5 | Perform operations with integers.
6 | Use the order of operations agreement.

2 The Integers; Order of Operations

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In 2009, the United States faced the worst economic crisis since the Great Depression. From the political left and right, watchdogs of the federal budget warned of continued fiscal trouble. President Obama's \$787 billion economic stimulus package of spending increases and tax cuts contributed to a record projected federal deficit of \$1.75 trillion. In this section, we use operations on a set of numbers called the *integers* to describe numerically the sad state of the nation's finances.

1 | Define the integers.

Defining the Integers

In Section 1, we applied some ideas of number theory to the set of natural, or counting, numbers:

$$\text{Natural numbers} = \{1, 2, 3, 4, 5, \dots\}.$$

When we combine the number 0 with the natural numbers, we obtain the set of **whole numbers**:

$$\text{Whole numbers} = \{0, 1, 2, 3, 4, 5, \dots\}.$$

The whole numbers do not allow us to describe certain everyday situations. For example, if the balance in your checking account is \$30 and you write a check for \$35, your checking account is overdrawn by \$5. We can write this as -5, read *negative* 5. The set consisting of the natural numbers, 0, and the negatives of the natural numbers is called the set of **integers**.

$$\text{Integers} = \{\underbrace{\dots, -4, -3, -2, -1}_{\substack{\text{Negative}\\ \text{integers}}}, 0, \underbrace{1, 2, 3, 4, \dots}_{\substack{\text{Positive}\\ \text{integers}}}\}$$

Notice that the term *positive integers* is another name for the natural numbers. The positive integers can be written in two ways:

1. Use a "+" sign. For example, +4 is "positive four."
2. Do not write any sign. For example, 4 is assumed to be "positive four."

The Number Line; The Symbols $<$ and $>$

The **number line** is a graph we use to visualize the set of integers, as well as sets of other numbers. The number line is shown in **Figure 1**.

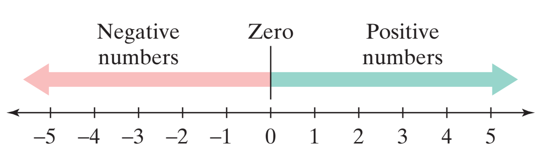


FIGURE 1 The number line

The number line extends indefinitely in both directions, shown by the arrows on the left and the right. Zero separates the positive numbers from the negative numbers on the number line. The positive integers are located to the right of 0 and the negative integers are located to the left of 0. **Zero is neither positive nor negative.** For every positive integer on a number line, there is a corresponding negative integer on the opposite side of 0.

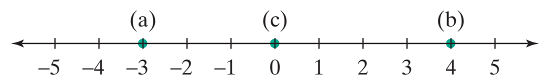
2 Graph integers on a number line.

Integers are graphed on a number line by placing a dot at the correct location for each number.

EXAMPLE 1 Graphing Integers on a Number Line

Graph: **a.** -3 **b.** 4 **c.** 0.

Solution Place a dot at the correct location for each integer.



CHECK POINT 1 Graph:

a. -4 **b.** 0 **c.** 3.

We will use the following symbols for comparing two integers:

$<$ means "is less than."

$>$ means "is greater than."

3 Use the symbols $<$ and $>$.

On the number line, the integers increase from left to right. The *lesser* of two integers is the one farther to the *left* on a number line. The *greater* of two integers is the one farther to the *right* on a number line.

Look at the number line in **Figure 2**. The integers -4 and -1 are graphed.

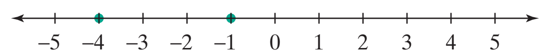


FIGURE 2

Observe that -4 is to the left of -1 on the number line. This means that -4 is less than -1.

$$-4 < -1$$

-4 is less than -1 because -4 is to the **left** of -1 on the number line.

In **Figure 2**, we can also observe that -1 is to the right of -4 on the number line. This means that -1 is greater than -4.

$$-1 > -4$$

-1 is greater than -4 because -1 is to the **right** of -4 on the number line.

The symbols $<$ and $>$ are called **inequality symbols**. These symbols always point to the lesser of the two real numbers when the inequality statement is true.

-4 is less than -1. $\quad -4 < -1 \quad$ The symbol points to -4, the lesser number.

-1 is greater than -4. $\quad -1 > -4 \quad$ The symbol still points to -4, the lesser number.

EXAMPLE 2 Using the Symbols $<$ and $>$

Insert either $<$ or $>$ in the shaded area between the integers to make each statement true:

a. -4 ■ 3 **b.** -1 ■ -5 **c.** -5 ■ -2 **d.** 0 ■ -3.

Solution The solution is illustrated by the number line in **Figure 3**.

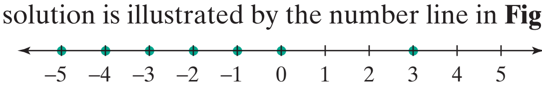


FIGURE 3

a. $-4 < 3$ (negative 4 is less than 3) because -4 is to the left of 3 on the number line.

STUDY TIP

You can think of negative integers as amounts of money that you *owe*. It's better to owe less, so

$$-1 > -5.$$

b. $-1 > -5$ (negative 1 is greater than negative 5) because -1 is to the right of -5 on the number line.

c. $-5 < -2$ (negative 5 is less than negative 2) because -5 is to the left of -2 on the number line.

d. $0 > -3$ (zero is greater than negative 3) because 0 is to the right of -3 on the number line.

CHECK POINT 2 Insert either $<$ or $>$ in the shaded area between the integers to make each statement true:

a. $6 \blacksquare -7$ **b.** $-8 \blacksquare -1$

c. $-25 \blacksquare -2$ **d.** $-14 \blacksquare 0$.

The symbols $<$ and $>$ may be combined with an equal sign, as shown in the following table:

Symbols	Meaning	Examples	Explanation
$a \le b$	*a* is less than or equal to *b*.	$2 \le 9$ $9 \le 9$	Because $2 < 9$ Because $9 = 9$
$b \ge a$	*b* is greater than or equal to *a*.	$9 \ge 2$ $2 \ge 2$	Because $9 > 2$ Because $2 = 2$

This inequality is true if either the $<$ part or the $=$ part is true.

This inequality is true if either the $>$ part or the $=$ part is true.

4 Find the absolute value of an integer.

Absolute Value

Absolute value describes distance from 0 on a number line. If *a* represents an integer, the symbol $|a|$ represents its absolute value, read "the absolute value of *a*." For example,

$$|-5| = 5.$$

The absolute value of -5 is 5 because -5 is 5 units from 0 on a number line.

ABSOLUTE VALUE

The **absolute value** of an integer *a*, denoted by $|a|$, is the distance from 0 to *a* on the number line. Because absolute value describes a distance, it is never negative.

EXAMPLE 3 Finding Absolute Value

Find the absolute value:

a. $|-3|$ **b.** $|5|$ **c.** $|0|$.

Solution The solution is illustrated in **Figure 4**.

a. $|-3| = 3$ The absolute value of -3 is 3 because -3 is 3 units from 0.

b. $|5| = 5$ 5 is 5 units from 0.

c. $|0| = 0$ 0 is 0 units from itself.

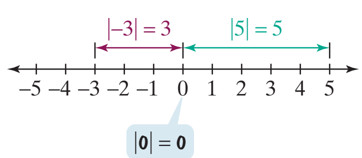


FIGURE 4 Absolute value describes distance from 0 on a number line.

Example 3 illustrates that the absolute value of a positive integer or 0 is the number itself. The absolute value of a negative integer, such as -3, is the number without the negative sign. Zero is the only real number whose absolute value is 0: $|0| = 0$. **The absolute value of any integer other than 0 is always positive.**

STUDY TIP

Do not confuse $|-3|$ with $-|3|$.

$$|-3| = 3 \qquad -|3| = -3$$

-3 is 3 units from 0.

The negative is not inside the absolute value bars and is not affected by the absolute value.

CHECK POINT 3 Find the absolute value:

a. $|-8|$ **b.** $|6|$. **c.** $-|8|$.

5 Perform operations with integers.

Addition of Integers

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It has not been a good day! First, you lost your wallet with \$30 in it. Then, you borrowed \$10 to get through the day, which you somehow misplaced. Your loss of \$30 followed by a loss of \$10 is an overall loss of \$40. This can be written

$$-30 + (-10) = -40.$$

The result of adding two or more numbers is called the **sum** of the numbers. The sum of -30 and -10 is -40.

You can think of gains and losses of money to find sums. For example, to find $17 + (-13)$, think of a gain of \$17 followed by a loss of \$13. There is an overall gain of \$4. Thus, $17 + (-13) = 4$. In the same way, to find $-17 + 13$, think of a loss of \$17 followed by a gain of \$13. There is an overall loss of \$4, so $-17 + 13 = -4$.

Using gains and losses, we can develop the following rules for adding integers:

RULES FOR ADDITION OF INTEGERS

Rule	Examples
If the integers have the same sign, **1.** Add their absolute values. **2.** The sign of the sum is the same as the sign of the two numbers.	$-11 + (-15) = -26$ Add absolute values: 11 + 15 = 26. Use the common sign.
If the integers have different signs, **1.** Subtract the smaller absolute value from the larger absolute value. **2.** The sign of the sum is the same as the sign of the number with the larger absolute value.	$-13 + 4 = -9$ Subtract absolute values: 13 − 4 = 9. Use the sign of the number with the greater absolute value. $13 + (-6) = 7$ Subtract absolute values: 13 − 6 = 7. Use the sign of the number with the greater absolute value.

TECHNOLOGY

Calculators and Adding Integers

You can use a calculator to add integers. Here are the keystrokes for finding $-11 + (-15)$:

Scientific Calculator

11 [+/−] [+] 15 [+/−] [=]

Graphing Calculator

[(−)] 11 [+] [(−)] 15 [ENTER]

Here are the keystrokes for finding $-13 + 4$:

Scientific Calculator

13 [+/−] [+] 4 [=]

Graphing Calculator

[(−)] 13 [+] 4 [ENTER]

STUDY TIP

In addition to gains and losses of money, another good analogy for adding integers is temperatures above and below zero on a thermometer. Think of the thermometer as a number line standing straight up. For example,

$-11 + (-15) = -26$ — If it's 11 below zero and the temperature falls 15 degrees, it will then be 26 below zero.

$-13 + 4 = -9$ — If it's 13 below zero and the temperature rises 4 degrees, the new temperature will be 9 below zero.

$13 + (-6) = 7.$ — If it's 13 above zero and the temperature falls 6 degrees, it will then be 7 above zero.

Using the analogies of gains and losses of money or temperatures can make the formal rules for addition of integers easy to use.

Can you guess what number is displayed if you use a calculator to find a sum such as $18 + (-18)$? If you gain 18 and then lose 18, there is neither an overall gain nor loss. Thus,

$$18 + (-18) = 0.$$

We call 18 and -18 **additive inverses**. Additive inverses have the same absolute value, but lie on opposite sides of zero on the number line. Thus, -7 is the additive inverse of 7, and 5 is the additive inverse of -5. In general, the sum of any integer and its additive inverse is 0:

$$a + (-a) = 0.$$

Subtraction of Integers

Suppose that a computer that normally sells for $1500 has a price reduction of $600. The computer's reduced price, $900, can be expressed in two ways:

$$1500 - 600 = 900 \quad \text{or} \quad 1500 + (-600) = 900.$$

This means that

$$1500 - 600 = 1500 + (-600).$$

To subtract 600 from 1500, we add 1500 and the additive inverse of 600. Generalizing from this situation, we define subtraction as follows:

DEFINITION OF SUBTRACTION

For all integers a and b,

$$a - b = a + (-b).$$

In words, to subtract b from a, add the additive inverse of b to a. The result of subtraction is called the **difference**.

TECHNOLOGY

Calculators and Subtracting Integers

You can use a calculator to subtract integers. Here are the keystrokes for finding $17 - (-11)$:

Scientific Calculator

17 [−] 11 [+/−] [=]

Graphing Calculator

17 [−] [(−)] 11 [ENTER]

Here are the keystrokes for finding $-18 - (-5)$:

Scientific Calculator

18 [+/−] [−] 5 [+/−] [=]

Graphing Calculator

[(−)] 18 [−] [(−)] 5 [ENTER]

Don't confuse the subtraction key on a graphing calculator, [−], with the sign change or additive inverse key, [(−)]. What happens if you do?

EXAMPLE 4 Subtracting Integers

Subtract:

a. $17 - (-11)$ **b.** $-18 - (-5)$ **c.** $-18 - 5$.

Solution

a. $17 - (-11) = 17 + 11 = 28$

Change the subtraction to addition. Replace -11 with its additive inverse.

b. $-18 - (-5) = -18 + 5 = -13$

Change the subtraction to addition. Replace -5 with its additive inverse.

c. $-18 - 5 = -18 + (-5) = -23$

Change the subtraction to addition. Replace 5 with its additive inverse.

CHECK POINT 4 Subtract:

a. $30 - (-7)$ **b.** $-14 - (-10)$ **c.** $-14 - 10$.

STUDY TIP

You can think of subtracting a negative integer as taking away a debt. Let's apply this analogy to $17 - (-11)$. Your checking account balance is $17 after an erroneous $11 charge was made against your account. When you bring this error to the bank's attention, they will take away the $11 debit and your balance will go up to $28:

$$17 - (-11) = 28.$$

Subtraction is used to solve problems in which the word *difference* appears. The difference between integers a and b is expressed as $a - b$.

EXAMPLE 5 An Application of Subtraction Using the Word *Difference*

The bar graph in **Figure 5** shows the budget surplus or deficit for the United States government from 2000 through 2008. What is the difference between the 2000 surplus and the 2008 deficit?

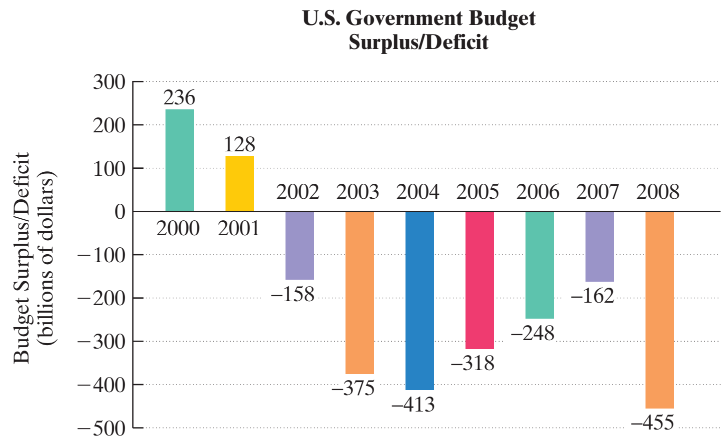

FIGURE 5
Source: Budget of the U.S. Government

Solution

The difference is the 2000 surplus minus the 2008 deficit.

$$= 236 - (-455)$$
$$= 236 + 455 = 691$$

The difference between the 2000 surplus and the 2008 deficit is $691 billion.

CHECK POINT 5 Use **Figure 5** to find the difference between the 2007 deficit and the 2008 deficit.

Multiplication of Integers

The result of multiplying two or more numbers is called the **product** of the numbers. You can think of multiplication as repeated addition or subtraction that starts at 0. For example,

$$3(-4) = 0 + (-4) + (-4) + (-4) = -12$$

The numbers have different signs and the product is negative.

and

$$(-3)(-4) = 0 - (-4) - (-4) - (-4) = 0 + 4 + 4 + 4 = 12.$$

The numbers have the same sign and the product is positive.