PEARSON NEW INTERNATIONAL EDITION

Calculus
Early Transcendentals

C. Henry Edwards David E. Penney

Seventh Edition

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In Problems 29 through 32, the graph of f and the x-axis divide the xy-plane into several regions, some of which are bounded. Find the total area of the bounded regions in each problem.

29. $f(x) = 1 - x^4$ if $x \leqq 0$; $f(x) = 1 - x^3$ if $x \geqq 0$ (Fig. 5.6.8)

30. $f(x) = (\pi/2)^2 \sin x$ on $[0, \pi/2]$; $f(x) = x(\pi - x)$ on $[\pi/2, \pi]$ (Fig. 5.6.9)

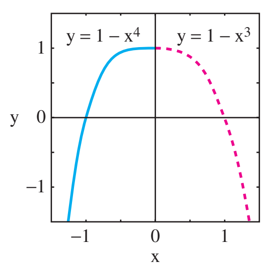

FIGURE 5.6.8 Problem 29.

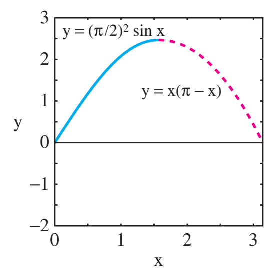

FIGURE 5.6.9 Problem 30.

31. $f(x) = x^3 - 9x$ (Fig. 5.6.10)

32. $f(x) = x^3 - 2x^2 - 15x$ (Fig. 5.6.11)

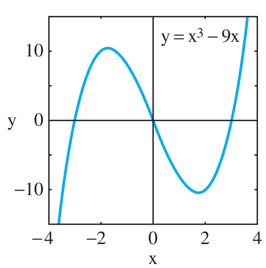

FIGURE 5.6.10 Problem 31.

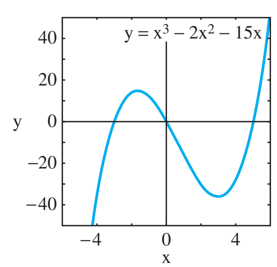

FIGURE 5.6.11 Problem 32.

33. Rosanne drops a ball from a height of 400 ft. Find the ball's average height and its average velocity between the time it is dropped and the time it strikes the ground.

34. Find the average value of the animal population $P(t) = 100 + 10t + (0.02)t^2$ over the time interval [0, 10].

35. Suppose that a 5000-L water tank takes 10 min to drain and that after t minutes, the amount of water remaining in the tank is $V(t) = 50(10 - t)^2$ liters. What is the average amount of water in the tank during the time it drains?

36. On a certain day the temperature t hours past midnight was

$$T(t) = 80 + 10\sin\left(\frac{\pi}{12}(t - 10)\right).$$

What was the average temperature between noon and 6 P.M.?

37. Suppose that a heated rod lies along the interval $0 \leqq x \leqq 10$. If the temperature at points of the rod is given by $T(x) = 4x(10 - x)$, what is the rod's average temperature?

38. Figure 5.6.12 shows a cross section at distance x from the center of a sphere of radius 1. Find the average area of the cross section for $0 \leqq x \leqq 1$.

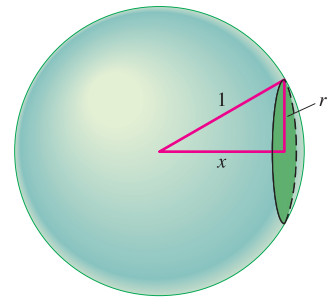

FIGURE 5.6.12 The sphere of Problem 38.

39. Figure 5.6.13 shows a cross section at distance y from the vertex of a cone with base radius 1 and height 2. Find the average area of this cross section for $0 \leqq y \leqq 2$.

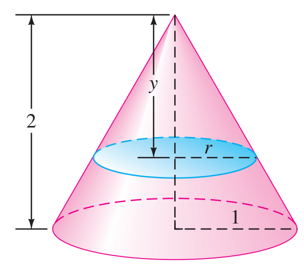

FIGURE 5.6.13 The cone of Problem 39.

40. A sports car starts from rest ($x = 0$, $t = 0$) and experiences constant acceleration $x''(t) = a$ for T seconds. Find, in terms of a and T, (a) its final and average velocities and (b) its final and average positions.

41. (a) Figure 5.6.14 shows a triangle inscribed in the region that lies between the x-axis and the curve $y = 9 - x^2$. Express the area of this triangle as a function $A(x)$ of the x-coordinate of its upper vertex P. (b) Find the average area $\overline{A}$ of $A(x)$ for x in the interval $[-3, 3]$. (c) Sketch a triangle as in Fig. 5.6.14 that has the area $\overline{A}$ found in part (b). How many different such triangles are there?

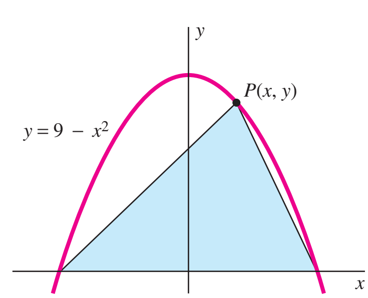

FIGURE 5.6.14 The typical triangle of Problem 41.

42. (a) Figure 5.6.15 shows a rectangle inscribed in the first-quadrant region that lies between the x-axis and the line $y = 10 - x$. Express the area of this rectangle as a function $A(x)$ of the x-coordinate of its vertex P on the line. (b) Find the average area $\overline{A}$ of $A(x)$ for x in the interval [0, 10]. (c) Sketch a rectangle as in Fig. 5.6.15 that has the area $\overline{A}$ found in part (b). How many different such rectangles are there?

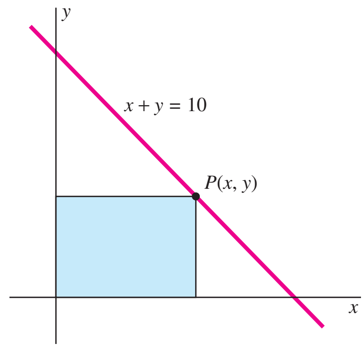

FIGURE 5.6.15 The typical rectangle of Problem 42.

43. (a) Figure 5.6.16 shows a rectangle inscribed in the semicircular region that lies between the x-axis and the graph $y = \sqrt{16 - x^2}$. Express the area of the rectangle as a function $A(x)$ of the x-coordinate of its vertex P on the line. (b) Find the average area $\overline{A}$ of $A(x)$ for x in the interval $[0, 4]$. (c) Sketch a rectangle as in Fig. 5.6.16 that has the area $\overline{A}$ found in part (b). How many different such rectangles are there?

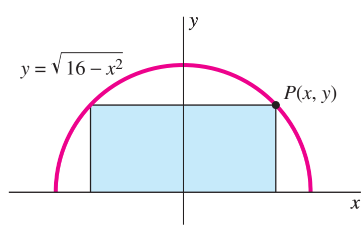

FIGURE 5.6.16 The typical rectangle of Problem 43.

44. Repeat Problem 43 in the case that the rectangle has two vertices on the x-axis and two on the parabola $y = 16 - x^2$ (rather than on the semicircle $y = \sqrt{16 - x^2}$). You may need to use a calculator or computer to find the base of a rectangle whose area is the average $\overline{A}$ of $A(x)$ for x in $[0, 4]$.

In Problems 45 through 49, apply the fundamental theorem of calculus to find the derivative of the given function.

45. $f(x) = \displaystyle\int_{-1}^{x} (t^2 + 1)^{17}\,dt$ **46.** $g(t) = \displaystyle\int_{0}^{t} \sqrt{x^2 + 25}\,dx$

47. $h(z) = \displaystyle\int_{2}^{z} \sqrt[3]{u - 1}\,du$ **48.** $A(x) = \displaystyle\int_{1}^{x} \frac{1}{t}\,dt$

49. $f(x) = \displaystyle\int_{x}^{10} (e^t - e^{-t})\,dt$

In Problems 50 through 53, $G(x)$ is the integral of the given function $f(t)$ over the specified interval of the form $[a, x]$, $x > a$. Apply Part 1 of the fundamental theorem of calculus to find $G'(x)$.

50. $f(t) = \dfrac{t}{t^2 + 1}$; $[2, x]$ **51.** $f(t) = \sqrt{t + 4}$; $[0, x]$

52. $f(t) = \sin^3 t$; $[0, x]$ **53.** $f(t) = \sqrt{t^3 + 1}$; $[1, x]$

In Problems 54 through 60, differentiate the function by first writing $f(x)$ in the form $g(u)$, where u denotes the upper limit of integration.

54. $f(x) = \displaystyle\int_{0}^{x^2} \sqrt{1 + t^3}\,dt$ **55.** $f(x) = \displaystyle\int_{2}^{3x} \sin t^2\,dt$

56. $f(x) = \displaystyle\int_{0}^{\sin x} \sqrt{1 - t^2}\,dt$ **57.** $f(x) = \displaystyle\int_{0}^{x^2} \sin t\,dt$

58. $f(x) = \displaystyle\int_{1}^{\sin x} (t^2 + 1)^3\,dt$ **59.** $f(x) = \displaystyle\int_{1}^{x^2+1} \frac{dt}{t}$

60. $f(x) = \displaystyle\int_{1}^{e^x} \ln(1 + t^2)\,dt$

Use integrals (as in Example 9) to solve the initial value problems in Problems 61 through 64.

61. $\dfrac{dy}{dx} = \dfrac{1}{x}$, $y(1) = 0$

62. $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$, $y(1) = \dfrac{\pi}{4}$

63. $\dfrac{dy}{dx} = \sqrt{1 + x^2}$, $y(5) = 10$

64. $\dfrac{dy}{dx} = \tan x$, $y(1) = 2$

65. The fundamental theorem of calculus *seems* to say that

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2,$$

in apparent contradiction to the fact that $1/x^2$ is always positive. What's wrong here?

66. Prove that the average rate of change

$$\frac{f(b) - f(a)}{b - a}$$

of the differentiable function f on $[a, b]$ is equal to the average value of its derivative on $[a, b]$.

67. The graph $y = f(x)$, $0 \leqq x \leqq 10$ is shown in Fig. 5.6.17. Let

$$g(x) = \int_{0}^{x} f(t)\,dt.$$

(a) Find the values $g(0)$, $g(2)$, $g(4)$, $g(6)$, $g(8)$, and $g(10)$. (b) Find the intervals on which $g(x)$ is increasing and those on which it is decreasing. (c) Find the global maximum and minimum values of $g(x)$ for $0 \leqq x \leqq 10$. (d) Sketch a rough graph of $y = g(x)$.

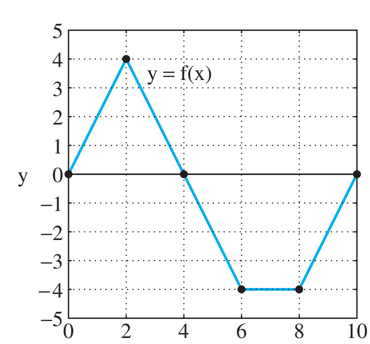

FIGURE 5.6.17 Problem 67.

68. Repeat Problem 67, except use the graph of the function f shown in Fig. 5.6.18.

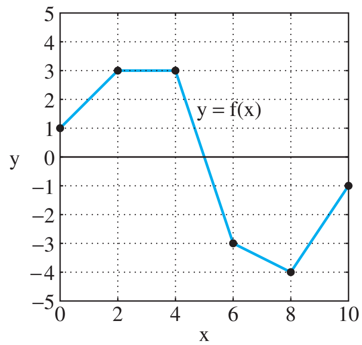

FIGURE 5.6.18 Problem 68.

69. Figure 5.6.19 shows the graph of the function $f(x) = x \sin x$ on the interval $[0, 4\pi]$. Let

$$g(x) = \int_0^x f(t)\,dt.$$

(a) Find the values of x at which $g(x)$ has local maximum and minimum values on the interval $[0, 4\pi]$. (b) Where does $g(x)$ attain its global maximum and minimum values on $[0, 4\pi]$? (c) Which points on the graph $y = f(x)$ correspond to inflection points on the graph $y = g(x)$? (d) Sketch a rough graph of $y = g(x)$.

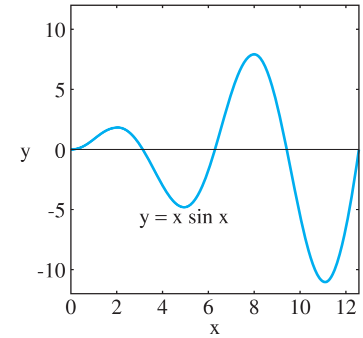

FIGURE 5.6.19 Problem 69.

70. Repeat Problem 69, except use the function

$$f(x) = \frac{\sin x}{x}$$

on the interval $[0, 4\pi]$ (as shown in Fig. 5.6.20). Take $f(0) = 1$ because $(\sin x)/x \to 1$ as $x \to 0$.

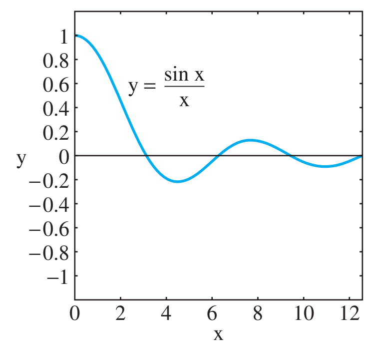

FIGURE 5.6.20 Problem 70.

5.7 INTEGRATION BY SUBSTITUTION

The fundamental theorem of calculus in the form

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b \tag{1}$$

implies that we can readily evaluate the definite integral on the left if we can find the indefinite integral (that is, antiderivative) on the right. We now discuss a powerful method of antidifferentiation that amounts to "the chain rule in reverse." This method is a generalization of the "generalized power rule in reverse,"

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1), \tag{2}$$

which we introduced in Section 5.2.

Equation (2) is an abbreviation for the formula

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C \quad (n \neq -1) \tag{3}$$

that results when we write

$$u = g(x), \qquad du = g'(x)\,dx.$$

So to apply Eq. (2) to a given integral, we must be able to visualize the integrand as a *product* of a power of a differentiable function $g(x)$ and its derivative $g'(x)$.

EXAMPLE 1 With

$$u = 2x + 1, \qquad du = 2\,dx,$$

we see that

$$\int (2x+1)^5 \cdot 2\,dx = \int u^5\,du = \frac{u^6}{6} + C = \frac{1}{6}(2x+1)^6 + C.$$

EXAMPLE 2

(a) $$\int 2x\sqrt{1+x^2}\,dx = \int (1+x^2)^{1/2} \cdot 2x\,dx$$
$$= \int u^{1/2}\,du \qquad (u = 1 + x^2, \quad du = 2x\,dx)$$
$$= \frac{u^{3/2}}{\frac{3}{2}} + C = \tfrac{2}{3}(1+x^2)^{3/2} + C.$$

(b) Similarly, but with $u = 1 + e^x$ and $du = e^x\,dx$, we get

$$\int \frac{e^x}{\sqrt{1+e^x}}\,dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C$$
$$= 2\sqrt{1+e^x} + C.$$

Equation (3) is the special case $f(u) = u^n$ of the general integral formula

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du. \qquad \textbf{(4)}$$

The right-hand side of Eq. (4) results when we make the formal substitutions

$$u = g(x), \qquad du = g'(x)\,dx$$

on the left-hand side.

One of the beauties of differential notation is that Eq. (4) is not only plausible but is, in fact, true—with the understanding that u is to be replaced with $g(x)$ after the indefinite integration on the right-hand side of Eq. (4) has been carried out. Indeed, Eq. (4) is merely an indefinite integral version of the chain rule. For if $F'(x) = f(x)$, then

$$D_x F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

by the chain rule, so

$$\int f(g(x)) \cdot g'(x)\,dx = \int F'(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$$
$$= F(u) + C \quad [u = g(x)]$$
$$= \int f(u)\,du.$$

Equation (4) is the basis for the powerful technique of indefinite **integration by substitution.** It may be used whenever the integrand function is recognized to be of the form $f(g(x)) \cdot g'(x)$.

EXAMPLE 3 Find

$$\int x^2\sqrt{x^3+9}\,dx.$$

Solution Note that x^2 is, to within a ***constant*** factor, the derivative of $x^3 + 9$. We can, therefore, substitute

$$u = x^3 + 9, \qquad du = 3x^2\,dx. \tag{5}$$

The constant factor 3 can be supplied if we compensate by multiplying the integral by $\frac{1}{3}$. This gives

$$\begin{aligned}\int x^2\sqrt{x^3+9}\,dx &= \frac{1}{3}\int (x^3+9)^{1/2}\cdot 3x^2\,dx = \frac{1}{3}\int u^{1/2}\,du \\ &= \frac{1}{3}\cdot\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{2}{9}u^{3/2} + C = \frac{2}{9}(x^3+9)^{3/2} + C.\end{aligned}$$

An alternative way to carry out the substitution in (5) is to solve

$$du = 3x^2\,dx \quad \text{for} \quad x^2\,dx = \tfrac{1}{3}\,du,$$

and then write

$$\int (x^3+9)^{1/2}\,dx = \int u^{1/2}\cdot\tfrac{1}{3}\,du = \tfrac{1}{3}\int u^{1/2}\,du,$$

concluding the computation as before.

The following three steps in the solution of Example 3 are worth special mention:

- The differential dx along with the rest of the integrand is "transformed," or replaced, in terms of u and du.
- Once the integration has been performed, the constant C of integration is added.
- A final resubstitution is necessary to write the answer in terms of the original variable x.

Substitution in Trigonometric and Exponential Integrals

By now we know that every differentiation formula yields—upon "reversal"—a corresponding antidifferentiation formula. The familiar formulas for the derivatives of the six trigonometric functions thereby yield the following indefinite-integral formulas:

$$\int \cos u\,du = \sin u + C, \tag{6}$$

$$\int \sin u\,du = -\cos u + C, \tag{7}$$

$$\int \sec^2 u\,du = \tan u + C, \tag{8}$$

$$\int \csc^2 u\,du = -\cot u + C, \tag{9}$$

$$\int \sec u\,\tan u\,du = \sec u + C, \tag{10}$$

$$\int \csc u\,\cot u\,du = -\csc u + C. \tag{11}$$

Also, the derivatives $D_x[e^x] = e^x$ and $D_x[\ln|x|] = 1/x$ (for $x \neq 0$) yield the integral formulas

$$\int e^u\,du = e^u + C, \tag{12}$$

$$\int \frac{1}{u}\,du = \ln|u| + C \quad (\text{for } u \neq 0). \tag{13}$$

Any of these integrals can appear as the integral $\int f(u)\,du$ that results from an appropriate *u-substitution* in a given integral.

EXAMPLE 4

$$\begin{aligned}\int \sin(3x+4)\,dx &= \int (\sin u)\cdot\tfrac{1}{3}\,du \qquad (u = 3x+4, \quad du = 3\,dx)\\ &= \tfrac{1}{3}\int \sin u\,du = -\tfrac{1}{3}\cos u + C\\ &= -\tfrac{1}{3}\cos(3x+4) + C.\end{aligned}$$

EXAMPLE 5

$$\begin{aligned}\int 3x\cos(x^2)\,dx &= 3\int (\cos x^2)\cdot x\,dx\\ &= 3\int (\cos u)\cdot\tfrac{1}{2}\,du \qquad (u = x^2, \quad du = 2x\,dx)\\ &= \tfrac{3}{2}\int \cos u\,du = \tfrac{3}{2}\sin u + C = \tfrac{3}{2}\sin(x^2) + C.\end{aligned}$$

EXAMPLE 6

$$\begin{aligned}\int \sec^2 3x\,dx &= \int (\sec^2 u)\cdot\tfrac{1}{3}\,du \qquad (u = 3x, \quad du = 3\,dx)\\ &= \tfrac{1}{3}\tan u + C = \tfrac{1}{3}\tan 3x + C.\end{aligned}$$

EXAMPLE 7 Evaluate

$$\int 2\sin^3 x\,\cos x\,dx.$$

Solution None of the integrals in Eqs. (6) through (11) appears to “fit,” but the substitution

$$u = \sin x, \qquad du = \cos x\,dx$$

yields

$$\int 2\sin^3 x\,\cos x\,dx = 2\int u^3\,du = 2\cdot\frac{u^4}{4} + C = \frac{1}{2}\sin^4 x + C.$$

EXAMPLE 8 Let

$$u = 1 + \sqrt{x^3} = 1 + x^{3/2}, \quad \text{so that} \quad du = \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}\sqrt{x}\,dx.$$

Then Eq. (12) yields

$$\int 3\sqrt{x}\,\exp\left(1+\sqrt{x^3}\right)dx = \int e^u\cdot 2\,du = 2e^u + C = 2\exp\left(1+\sqrt{x^3}\right) + C.$$