PEARSON NEW INTERNATIONAL EDITION

Basic Technical Mathematics with Calculus

Allyn J. Washington

Tenth Edition

PEARSON

Pearson New International Edition

Basic Technical Mathematics
with Calculus
Allyn J. Washington
Tenth Edition

PEARSON

EXAMPLE 2 Case 1: Two angles and one side

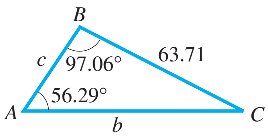

Fig. 54

Solve the triangle with the following given parts: $a = 63.71$, $A = 56.29°$, and $B = 97.06°$. See Fig. 54.

From the figure, we see that we are to find angle C and sides b and c. We first determine angle C:

$$\begin{aligned} C &= 180° - (A + B) = 180° - (56.29° + 97.06°) \\ &= 26.65° \end{aligned}$$

Noting the three angles, we know that c is the shortest side (C is the smallest angle) and b is the longest side (B is the largest angle). This means that the length of a is between c and b, or $c < 63.71$ and $b > 63.71$. Now using the ratio $a/\sin A$ of Eq. (8) (the law of sines) to find sides b and c, we have

$$\frac{b}{\underset{B}{\sin 97.06°}} = \frac{\overset{a}{63.71}}{\underset{A}{\sin 56.29°}} \quad \text{or} \quad b = \frac{63.71 \sin 97.06°}{\sin 56.29°} = 76.01$$

$$\frac{c}{\underset{C}{\sin 26.65°}} = \frac{\overset{a}{63.71}}{\underset{A}{\sin 56.29°}} \quad \text{or} \quad c = \frac{63.71 \sin 26.65°}{\sin 56.29°} = 34.35$$

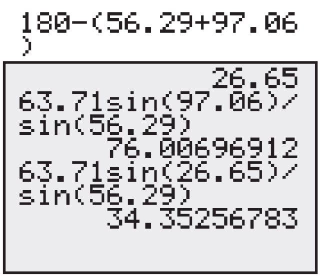

Fig. 55

Thus, $b = 76.01$, $c = 34.35$, and $C = 26.65°$. Note that $c < a$ and $b > a$, as expected. The calculator solution is shown in Fig. 55. ■

If the given information is appropriate, the law of sines may be used to solve applied problems. The following example illustrates the use of the law of sines in such a problem.

EXAMPLE 3 Case 1: Application

■ The first successful helicopter was made in the United States by Igor Sikorsky in 1939.

Two observers A and B sight a helicopter due east. The observers are 3540 ft apart, and the angles of elevation they each measure to the helicopter are 32.0° and 44.0°, respectively. How far is observer A from the helicopter? See Fig. 56.

Letting H represent the position of the helicopter, we see that angle B within the triangle ABH is $180° - 44.0° = 136.0°$. This means that the angle at H within the triangle is

$$H = 180° - (32.0° + 136.0°) = 12.0°$$

Now, using the law of sines to find required side b, we have

required side opposite known angle → $\dfrac{b}{\sin 136.0°} = \dfrac{3540}{\sin 12.0°}$ ← known side opposite known angle

or

$$b = \frac{3540 \sin 136.0°}{\sin 12.0°} = 11{,}800 \text{ ft}$$

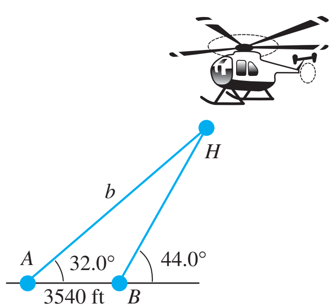

Fig. 56

Thus, observer A is about 11,800 ft from the helicopter. ■

CASE 2: TWO SIDES AND THE ANGLE OPPOSITE ONE OF THEM

For a triangle in which we know two sides and the angle opposite one of the given sides, the solution will be either *one triangle*, or *two triangles*, or even possibly *no triangle*. The following examples illustrate how each of these results is possible.

EXAMPLE 4 Case 2: Two sides and angle opposite

Solve the triangle with the following given parts: $a = 60.0$, $b = 40.0$, and $B = 30.0°$.

First, make a good scale drawing (Fig. 57(a)) by drawing angle B and measuring off 60 for a. This will more clearly show that side $b = 40.0$ will intersect side c at either position A or A'. This means there are two triangles that satisfy the given values. Using the law of sines, we solve the case for which A is an acute angle:

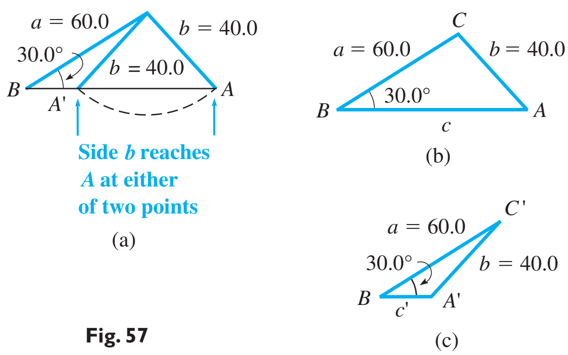

Fig. 57

$$\frac{60.0}{\sin A} = \frac{40.0}{\sin 30.0°} \quad \text{or} \quad \sin A = \frac{60.0 \sin 30.0°}{40.0}$$

$$A = \sin^{-1}\left(\frac{60.0 \sin 30.0°}{40.0}\right) = 48.6°$$

$$C = 180° - (30.0° + 48.6°) = 101.4°$$

Therefore, $A = 48.6°$ and $C = 101.4°$. Using the law of sines again to find c, we have

$$\frac{c}{\sin 101.4°} = \frac{40.0}{\sin 30.0°}$$

$$c = \frac{40.0 \sin 101.4°}{\sin 30.0°}$$

$$= 78.4$$

Thus, $A = 48.6°$, $C = 101.4°$, and $c = 78.4$. See Fig. 57(b).

The other solution is the case in which A', opposite side a, is an obtuse angle. Therefore,

$$A' = 180° - A = 180° - 48.6°$$

$$= 131.4°$$

$$C' = 180° - (30.0° + 131.4°)$$

$$= 18.6°$$

Using the law of sines to find c', we have

$$\frac{c'}{\sin 18.6°} = \frac{40.0}{\sin 30.0°}$$

$$c' = \frac{40.0 \sin 18.6°}{\sin 30.0°}$$

$$= 25.5$$

This means that the second solution is $A' = 131.4°$, $C' = 18.6°$, and $c' = 25.5$. See Fig. 57(c).

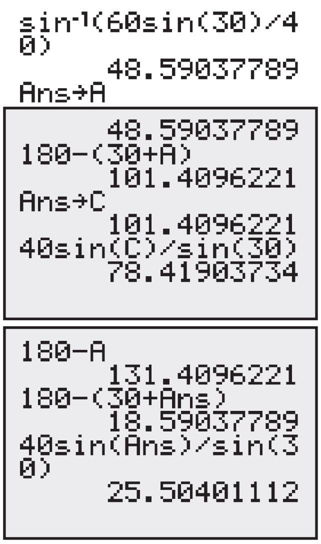

Fig. 58

The complete sequence for the calculator solution is shown in Fig. 58. The upper window shows the completion of the solution for A, C, and c. The lower window shows the solution for A', C', and c'. ■

EXAMPLE 5 Case 2: Possible solutions

In Example 4, if $b > 60.0$, only one solution would result. In this case, side b would intercept side c at A. It also intercepts the extension of side c, but this would require that angle B not be included in the triangle (see Fig. 59). Thus, only one solution may result if $b > a$.

In Example 4, there would be *no solution* if side b were not at least 30.0. If this were the case, side b would not be long enough to even touch side c. It can be seen that b must at least equal $a \sin B$. If it is just equal to $a \sin B$, there is *one solution,* a right triangle. See Figure 60.

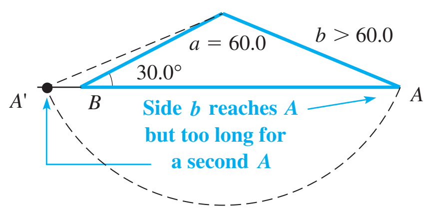

Fig. 59

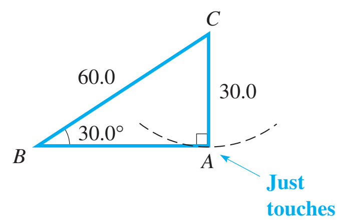

Fig. 60

Ambiguous Case

Summarizing the results for Case 2 as illustrated in Examples 4 and 5, we make the following conclusions. Given sides a and b and angle A (assuming here that a and A ($A < 90°$) are corresponding parts), we have the following summary of solutions for Case 2.

Practice Exercise

2. Determine which of the four possible solution types occurs if $c = 28$, $b = 48$, and $C = 30°$.

CAUTION

Summary of Solutions:
Two Sides and the Angle Opposite One of Them

1. **No solution** *if* $a < b \sin A$. See Fig. 61(a).
2. **A right triangle solution** *if* $a = b \sin A$. See Fig. 61(b).
3. **Two solutions** *if* $b \sin A < a < b$. See Fig. 61(c).
4. **One solution** *if* $a > b$. See Fig. 61(d).

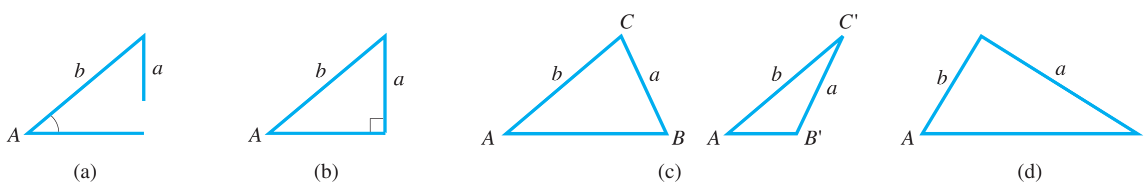

Fig. 61

NOTE

Note that *in order to have two solutions, we must know two sides and the angle opposite one of the sides, and the shorter side must be opposite the known angle.*

If there is *no solution,* the calculator will indicate an *error*. If the solution is a *right triangle,* the calculator will show an angle of *exactly* 90° (no extra decimal digits will be displayed).

NOTE

For the reason that *two solutions may result* for Case 2, it is called the **ambiguous case**. However, it must also be kept in mind that *there may only be one solution*. A *careful check of the given parts* must be made in order to determine whether there is one solution or two solutions. The following example illustrates Case 2 in an applied problem.

EXAMPLE 6 Case 2: Application

■ See the text introduction.

Kingston, Jamaica, is 43.2° south of east of Havana, Cuba. What should be the heading of a plane from Havana to Kingston if the wind is from the west at 40.0 km/h and the plane's speed with respect to the air is 300 km/h?

The heading should be set so that the resultant of the plane's velocity with respect to the air $\mathbf{v}_{pa}$ and the velocity of the wind $\mathbf{v}_w$ will be in the direction from Havana to Kingston. This means that the resultant velocity $\mathbf{v}_{pg}$ of the plane with respect to the ground must be at an angle of 43.2° south of east from Havana.

Using the given information, we draw the vector triangle shown in Fig. 62. In the triangle, we know that the angle at Kingston is 43.2° by noting the alternate-interior angles. By finding θ, the required heading can be found. There can be only one solution, because $v_{pa} > v_w$. Using the law of sines, we have

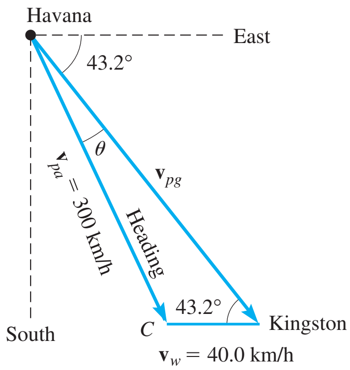

Fig. 62

known side opposite required angle → $\dfrac{40.0}{\sin\theta} = \dfrac{300}{\sin 43.2°}$ ← known side opposite known angle

$$\sin\theta = \frac{40.0 \sin 43.2°}{300}, \qquad \theta = 5.2°$$

Therefore, the heading should be $43.2° + 5.2° = 48.4°$ south of east. ■

Practice Exercise

3. In Example 6, what should be the heading if 300 km/h is changed to 500 km/h?

If we try to use the law of sines for Case 3 or Case 4, we find that we do not have enough information to complete any of the ratios. These cases can, however, be solved by the law of cosines as shown in the next section.

EXAMPLE 7 Cases 3&4 not solvable by law of sines

Given (case 3) two sides and the included angle of a triangle $a = 2, b = 3, C = 45°$, and (case 4) the three sides (case 4) $a = 5, b = 6, c = 7$, we set up the ratios

$$(\text{case } 3)\ \frac{2}{\sin A} = \frac{3}{\sin B} = \frac{c}{\sin 45°}, \text{ and } (\text{case } 4)\ \frac{5}{\sin A} = \frac{6}{\sin B} = \frac{7}{\sin C}$$

The solution cannot be found because each of the three possible equations in either case 3 or case 4 contains two unknowns. ■

EXERCISES 5

In Exercises 1 and 2, solve the resulting triangles if the given changes are made in the indicated examples of this section.

1. In Example 2, solve the triangle if the value of B is changed to 82.94°.
2. In Example 4, solve the triangle if the value of a is changed to 70.0.

In Exercises 3–22, solve the triangles with the given parts.

3. $a = 45.7, A = 65.0°, B = 49.0°$
4. $b = 3.07, A = 26.0°, C = 120.0°$
5. $c = 4380, A = 37.4°, B = 34.6°$
6. $a = 932, B = 0.9°, C = 82.6°$
7. $a = 4.601, b = 3.107, A = 18.23°$
8. $b = 362.2, c = 294.6, B = 110.63°$
9. $b = 7751, c = 3642, B = 20.73°$
10. $a = 150.4, c = 250.9, C = 76.43°$
11. $b = 0.0742, B = 51.0°, C = 3.4°$
12. $c = 729, B = 121.0°, C = 44.2°$
13. $a = 63.8, B = 58.4°, C = 22.2°$
14. $a = 0.130, A = 55.2°, B = 117.5°$
15. $b = 4384, B = 47.43°, C = 64.56°$
16. $b = 283.2, B = 13.79°, C = 103.62°$
17. $a = 5.240, b = 4.446, B = 48.13°$
18. $a = 89.45, c = 37.36, C = 15.62°$
19. $b = 2880, c = 3650, B = 31.4°$
20. $a = 0.841, b = 0.965, A = 57.1°$
21. $a = 450, b = 1260, A = 64.8°$
22. $a = 20, c = 10, C = 30°$

In Exercises 23–40, use the law of sines to solve the given problems.

23. A small island is approximately a triangle in shape. If the longest side of the island is 520 m, and two of the angles are 45° and 55°, what is the length of the shortest side?

24. A boat followed a triangular route going from dock A, to dock B, to dock C, and back to dock A. The angles turned were 135° at B and 125° at C. If B is 875 m from A, how far is it from B to C?

25. The loading ramp at a delivery service is 12.5 ft long and makes a 18.0° angle with the horizontal. If it is replaced with a ramp 22.5 ft long, what angle does the new ramp make with the horizontal?

26. In an aerial photo of a triangular field, the longest side is 86.0 cm, the shortest side is 52.5 cm, and the largest angle is 82.0°. The scale is 1 cm = 2 m. Find the actual length of the third side of the field.

27. The Pentagon (headquarters of the U.S. Department of Defense) is the largest office building in the world. It is a regular pentagon (five sides), 921 ft on a side. Find the greatest straight-line distance from one point on the outside of the building to another outside point (the length of a diagonal).

28. Two ropes hold a 175-lb crate as shown in Fig. 63. Find the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ in the ropes. (*Hint:* Move vectors so that they are tail to head to form a triangle. The vector sum $\mathbf{T}_1 + \mathbf{T}_2$ must equal 175 lb for equilibrium.)

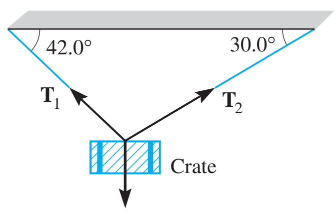

Fig. 63

29. Find the tension $\mathbf{T}$ in the left guy wire attached to the top of the tower shown in Fig. 64. (*Hint:* The horizontal components of the tensions must be equal and opposite for equilibrium. Thus, move the tension vectors tail to head to form a triangle with a vertical resultant. This resultant equals the upward force at the top of the tower for equilibrium. This last force is not shown and does not have to be calculated.)

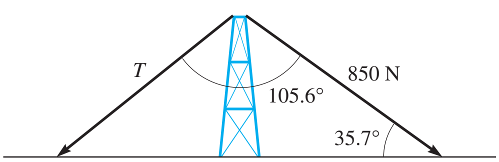

Fig. 64

30. Find the distance from Atlanta to Raleigh, North Carolina, from Fig. 65.

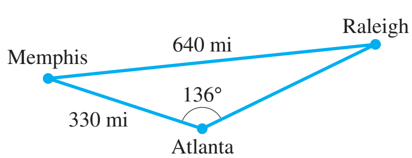

Fig. 65

31. Find the distance between Gravois Ave. and Jefferson Ave. along Arsenal St. in St. Louis, from Fig. 66.

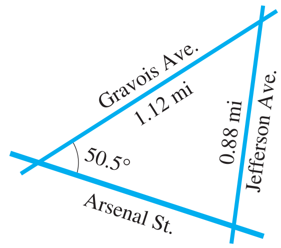

Fig. 66

32. When an airplane is landing at an 8250-ft runway, the angles of depression to the ends of the runway are 10.0° and 13.5°. How far is the plane from the near end of the runway?

33. Find the total length of the path of the laser beam that is shown in Fig. 67.

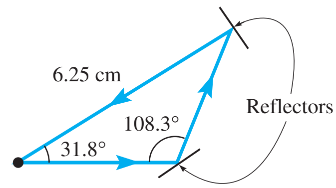

Fig. 67

34. In widening a highway, it is necessary for a construction crew to cut into the bank along the highway. The present angle of elevation of the straight slope of the bank is 23.0°, and the new angle is to be 38.5°, leaving the top of the slope at its present position. If the slope of the present bank is 220 ft long, how far horizontally into the bank at its base must they dig?

35. A communications satellite is directly above the extension of a line between receiving towers A and B. It is determined from radio signals that the angle of elevation of the satellite from tower A is 89.2°, and the angle of elevation from tower B is 86.5°. See Fig. 68. If A and B are 1290 km apart, how far is the satellite from A? (Neglect the curvature of the earth.)

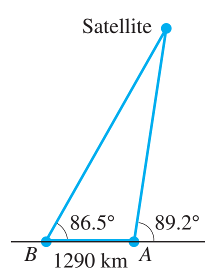

Fig. 68

36. An astronaut on the moon drives a lunar rover 16 km in the direction 60.0° north of east from the base. (a) Through what angle must the rover then be turned so that by driving 12 km farther the astronaut can turn again to return to base along a north-south line? (b) How long is the last leg of the trip? (c) Can the astronaut make it back to base if the maximum range of the rover is 40 km?

37. A boat owner wishes to cross a river 2.60 km wide and go directly to a point on the opposite side 1.75 km downstream. The boat goes 8.00 km/h in still water, and the stream flows at 3.50 km/h. What should the boat's heading be?

38. A motorist traveling along a level highway at 75 km/h directly toward a mountain notes that the angle of elevation of the mountain top changes from about 20° to about 30° in a 20-min period. How much closer on a direct line did the mountain top become?

39. A hillside is inclined at 23° with the horizontal. From a given point on the slope, it has been found that a vein of gold is 55 m directly below. At what angle below the hillside slope from another point downhill must a straight 65-m shaft be dug to reach the vein?

40. Point P on the mechanism shown in Fig. 69 is driven back and forth horizontally. If the minimum value of angle θ is 32.0°, what is the distance between extreme positions of P? What is the maximum possible value of angle θ?

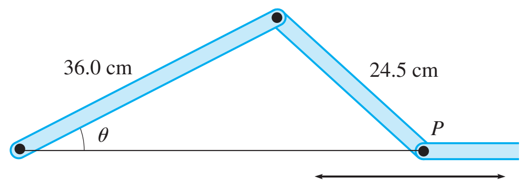

Fig. 69

Answers to Practice Exercises

1. $a = 6.34$ **2.** Two solutions **3.** 46.3°

6 The Law of Cosines

Law of Cosines • Case 3: Two Sides & Included Angle • Case 4: Three Sides • Summary of Solving Oblique Triangles

As noted in the last section, the law of sines cannot be used for Case 3 (two sides and the included angle) and Case 4 (three sides). In this section, we develop the *law of cosines,* which can be used for Cases 3 and 4. After finding another part of the triangle using the law of cosines, we will often find it easier to complete the solution using the law of sines.

Consider any oblique triangle—for example, either triangle shown in Fig. 70. For each triangle, $h/b = \sin A$, or $h = b \sin A$. Also, using the Pythagorean theorem, we obtain $a^2 = h^2 + x^2$ for each triangle. Therefore (with $(\sin A)^2 = \sin^2 A$),

$$\mathbf{a^2 = b^2 \sin^2 A + x^2} \tag{9}$$

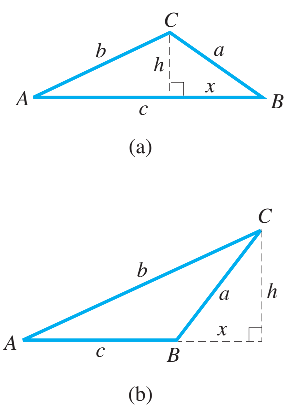

Fig. 70

In Fig. 70(a), note that $(c - x)/b = \cos A$, or $c - x = b \cos A$. Solving for x, we have $x = c - b \cos A$. In Fig. 70(b), $c + x = b \cos A$, and solving for x, we have $x = b \cos A - c$. Substituting these relations into Eq. (9), we obtain

$$\begin{aligned} &\mathbf{a^2 = b^2 \sin^2 A + (c - b \cos A)^2} \\ \text{and} \quad &\mathbf{a^2 = b^2 \sin^2 A + (b \cos A - c)^2} \end{aligned} \tag{10}$$

respectively. When expanded, these both give

$$\begin{aligned} \mathbf{a^2} &\mathbf{= b^2 \sin^2 A + b^2 \cos^2 A + c^2 - 2bc \cos A} \\ &\mathbf{= b^2(\sin^2 A + \cos^2 A) + c^2 - 2bc \cos A} \end{aligned} \tag{11}$$

Recalling the definitions of the trigonometric functions, we know that $\sin\theta = y/r$ and $\cos\theta = x/r$. Thus, $\sin^2\theta + \cos^2\theta = (y^2 + x^2)/r^2$. However, $x^2 + y^2 = r^2$, which means

$$\mathbf{\sin^2\theta + \cos^2\theta = 1} \tag{12}$$

This equation is valid for any angle θ, since we have made no assumptions as to the properties of θ. Thus, by substituting Eq. (12) into Eq. (11), we arrive at the **law of cosines:**

Law of Cosines

$$\boxed{a^2 = b^2 + c^2 - 2bc \cos A} \tag{13}$$

Using the method above, we may also show that

$$\boxed{\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}}$$