Pearson New International Edition

Excursions in Modern Mathematics

Peter Tannenbaum

Eighth Edition

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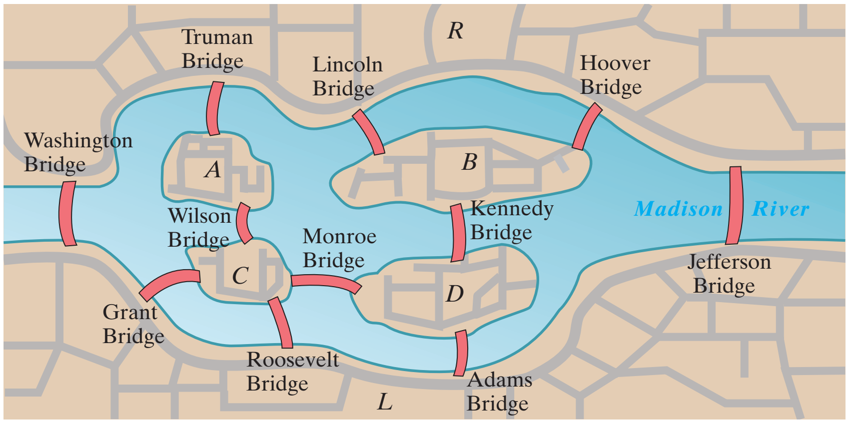


FIGURE 4 Bridges on the Madison River.

2 An Introduction to Graphs

A note of warning: The graphs we will be discussing here have no relation to the graphs of functions you may have studied in algebra or calculus.

The key tool we will use to tackle the street-routing problems introduced in Section 1 is the notion of a **graph**. The most common way to describe a *graph* is by means of a picture. The basic elements of such a picture are a set of "dots" called the **vertices** of the graph and a collection of "lines" called the **edges** of the graph. (Unfortunately, this terminology is not universal. In some applications the word "nodes" is used for the vertices and the word "links" is used for the edges. We will stick to vertices and edges as much as possible.) On the surface, that's all there is to it—edges connecting vertices. Below the surface there is a surprisingly rich theory. Let's look at a few examples first.

EXAMPLE 6 BASIC GRAPH CONCEPTS

Figure 5 shows several examples of graphs. We will discuss each separately.

- Figure 5(a) shows a graph with six vertices labeled *A, B, C, D, E,* and *F* (it is customary to use capital letters to label the vertices of a graph). For convenience we refer to the set of vertices of a graph as the **vertex set**. In this graph, the vertex set is $\{A, B, C, D, E, F\}$. The graph has 11 edges (described by listing, in any order, the two vertices that are connected by the edge): AB, AD, BC, etc.
  - When two vertices are connected by an edge we say that they are **adjacent vertices**. Thus, A and B are adjacent vertices, but A and E are not adjacent. The edge connecting B with itself is written as BB and is called a **loop**. Vertices C and D are connected twice (i.e., by two separate edges), so when we list the edges we include CD twice. Similarly, vertices E and F are connected by three edges, so we list EF three times. We refer to edges that appear more than once as **multiple edges**.
  - The complete list of edges of the graph, the **edge list**, is AB, AD, BB, BC, BE, CD, CD, DE, EF, EF, EF.
  - The number of edges that meet at each vertex is called the **degree** of the vertex and is denoted by $\deg(X)$. In this graph we have $\deg(A) = 2$, $\deg(B) = 5$ (please note that the loop contributes 2 to the degree of the vertex), $\deg(C) = 3$, $\deg(D) = 4$, $\deg(E) = 5$, and $\deg(F) = 3$. It will be important in the next section to distinguish between vertices depending on whether their degree is an odd or an even number. We will refer to vertices like B, C, E, and F with an odd degree as **odd vertices** and to vertices with an even degree like A and D as **even vertices**.
- Figure 5(b) is very similar to Fig. 5(a)—the only difference is the way the edge BE is drawn. In Fig. 5(a) edges AD and BE cross each other, but the crossing

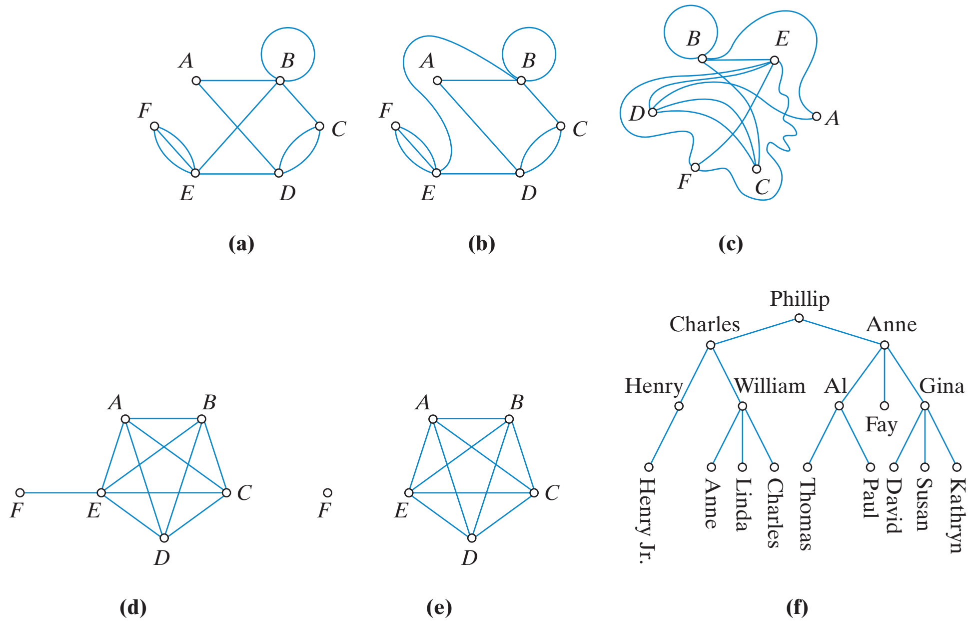
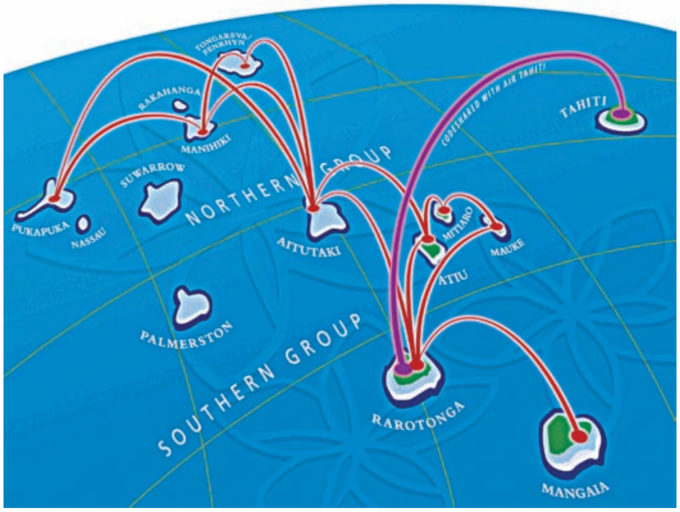


FIGURE 5 (a), (b), and (c) are all pictures of the same graph; (d) is a simple, connected graph; (e) is a simple, disconnected graph; and (f) is a graph labeled with names instead of letters.

point is not a vertex of the graph—it's just an irrelevant crossing point. Fig. 5(b) gets around the crossing by drawing the edge in a more convoluted way, but the way we draw an edge is itself irrelevant. The key point here is that as graphs, Figs. 5(a) and 5(b) are the same. Both have exactly the same vertices and exactly the same edge list.

- Figure 5(c) take the idea one step further—it is in fact, another rendering of the graph shown in Figs. 5(a) and (b). The vertices have been moved around and put in different positions, and the edges are funky—no other way to describe it. Despite all the funkiness, this graph conveys exactly the same information that the graph in Fig. 5(a) does. You can check it out— same set of vertices and same edge list. The moral here is that while graphs are indeed pictures connecting "dots" with "lines," it is not the specific picture that matters but the story that the picture tells: which dots are connected to each other and which aren't. We can move the vertices around, and we can draw the edges any funky way we want (straight line, curved line, wavy line, etc.)—none of that matters. The only thing that matters is the set of vertices and the list of edges.
- Figure 5(d) shows a graph with six vertices. Vertices *A*, *B*, *C*, *D*, and *E* form what is known as a **clique**—each vertex is connected to each of the other four. Vertex *F*, on the other hand, is connected to only one other vertex. This graph has no loops or multiple edges. Graphs without loops or multiple edges are called **simple graphs**. There are many applications of graphs where loops and multiple edges cannot occur, and we have to deal only with simple graphs. (In Examples 7 and 8 we will see two applications where only simple graphs occur.)
- Figure 5(e) shows a graph very similar to the one in Fig. 5(d). The only difference between the two is the absence of the edge *EF*. In this graph there are no edges connecting *F* to any other vertex. For obvious reasons, *F* is called an **isolated** vertex. This graph is made up of two separate and disconnected "pieces"—the clique formed by the vertices *A*, *B*, *C*, *D*, and *E* and the isolated vertex *F*. Because the graph is not made of a single "piece," we say that the graph is **disconnected**, and the separate pieces that make up the graph are called the *components* of the graph.
- Figure 5(f) shows a connected simple graph. The vertices of this graph are names (there is no rule about what the labels of a vertex can be). Can you guess what this graph might possibly represent?

EXAMPLE 7 AIRLINE ROUTE MAPS



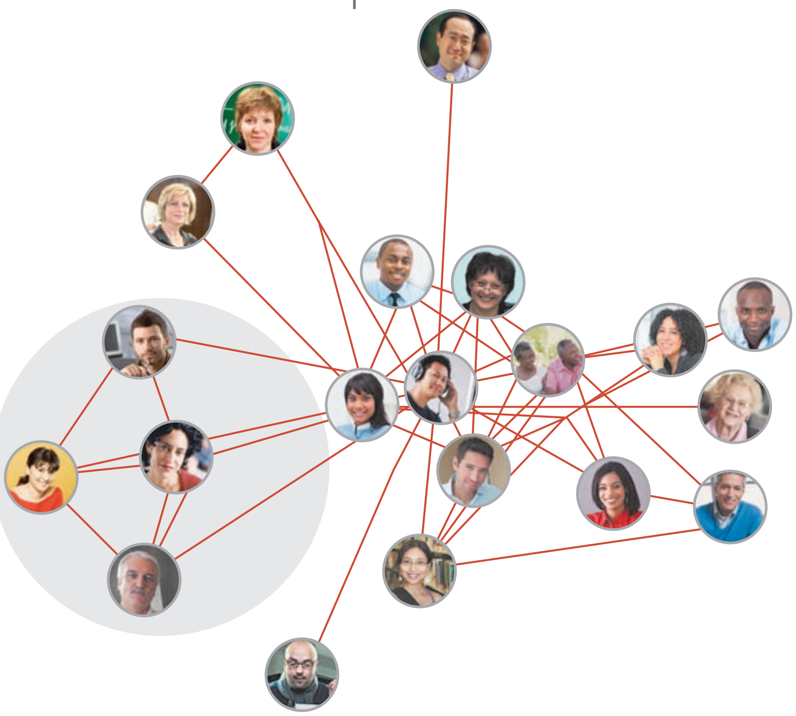
Ewan Smith/Air Rarotonga

FIGURE 6 Air Rarotonga route map.

Figure 6 shows the route map for a very small airline called Air Rarotonga. Air Rarotonga serves just 10 islands in the South Pacific, and the route map shows the direct flights that are available between the various islands. In essence, the route map is a graph whose vertices are the islands. An edge connects two islands if there is a direct flight between them. No direct flight, no edge. The picture makes a slight attempt to respect geographical facts (the bigger islands are drawn larger but certainly not to scale, and they sit in the ocean more or less as shown), but the point of an airline route map is to show if there is a direct flight from point X to point Y, and, in that regard, accurate geography is not all that important.

EXAMPLE 8 THE FACEBOOK SOCIAL GRAPH

Do you know who your friends' friends are? In the off-line world—where friendships are tight and relationships are personal—you probably have some idea. In the Facebook world—where "friendships" are cheap—you probably have a fairly limited picture of the complex web of friendships that connect your own set of Facebook friends.



Social network images: Shutterstock, Thinkstock, and Fotolia

FIGURE 7 The social graph: The friendship connections among your friends.

But worry no more. Thanks to the Facebook Social Graph (FSG), you can have a complete picture of how your Facebook friends are connected. The FSG is a Facebook app available for free to anyone on Facebook. [To create your own FSG (and you are strongly encouraged to do it now, not just for the curiosity factor but because it will help you navigate this example) go to your Facebook account, click on "Apps and Games," find "Social Graph" (you may have to do a search for it) and click "Allow."]

The vertices of the FSG are all your Facebook friends. You are not included because it is understood that you are a friend of everyone on the graph. An edge connecting two of your friends means that they are friends. Fig. 7 shows the FSG of a real person (who will remain nameless), but the FSG is at its most useful when used in a dynamic way. When you run the mouse over a particular friend X, the app will highlight all the connections between X and your other friends. Some X's are "hubs" connected to many people (if you are married it is very likely that your spouse is a hub, since you typically share most of your friends); other X's might be isolated vertices (the guy sitting next to you on the airplane with whom you exchanged pleasantries and ended up being a Facebook friend). The FSG also highlights "clusters" of friendships. These clusters represent groups of individuals who are all friends with each other (your high school buddies perhaps, or coworkers, or family).

The main point of Examples 7 and 8 is to highlight how powerful (and useful) the concept of a graph can be. Granted, the Air Rarotonga route map is small (just right to illustrate the point), but if you think big you can imagine a United Airlines route map instead, with hundreds of destinations and thousands of flights connecting them. The fundamental idea is still the same—graphs convey visually a tremendous amount of information that would be hard to convey in any other form. Can you imagine describing the complex web of relationships in your Facebook Social Graph or in a United Airlines route map any other way? Airline route maps and friendship graphs are always simple graphs, without loops (airlines don't routinely schedule flights that go around in circles, and by definition, friendship is a connection between two different persons) or multiple edges (either there are direct flights connecting *X* and *Y* or there aren't, and *X* and *Y* are either friends or they aren't).

EXAMPLE 9 PATHS AND CIRCUITS

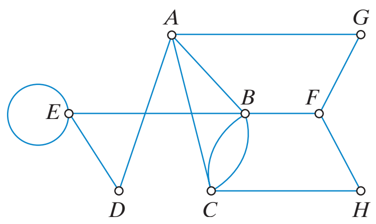


FIGURE 8 Graph for Example 9.

We say that two edges are **adjacent edges** when they share a common vertex. In Fig. 8 for example, *AB* is adjacent to *AC* and *AD* (they share vertex *A*), as well as to *BC*, *BF*, and *BE* (they share vertex *B*). A sequence of *distinct* edges each adjacent to the next is called a **path**, and the number of edges in the path is called the **length** of the path. For example in Fig. 8, the edges *AB*, *BF*, and *FG* form a path of length 3. A good way to think of a path is as a real-world path—a way to "hike" along the edges of the graph, traveling along the first edge, then the next, and so on. To shorten the notation, we describe the path by just listing the vertices in sequence separated by commas. For example *A*, *B*, *F*, *G* describes the path formed by the edges *AB*, *BF*, and *FG*.

Here are a few more examples of paths in Fig. 8:

- *A, B* is a path of length 1. Any edge can be thought of as a path of length 1—not very interesting, but it allows us to apply the concept of a path even to single edges.
- *A*, *B*, *C*, *A*, *D*, *E* is a path of length 5 starting at *A* and ending at *E*. The path goes through vertex *A* a second time, but that's OK. It is permissible for a path to revisit some of the vertices. On the other hand, *A, C, B, A, C* does not meet the definition of a path because the edges of the path cannot be revisited and here *AC* is traveled twice. So, in a path it's OK to revisit some of the vertices but not OK to revisit any edges.
- *A*, *B*, *C*, *A*, *D*, *E, E, B* is a path of length 7. Notice that this "trip" is possible because of the loop at *E*.
- *A*, *B*, *C*, *A*, *D*, *E, B, C* is also a legal path of length 7. Here we can use the edge *BC* twice because there are in fact two distinct edges connecting *B* and *C*.

When a trip along the edges of the graph closes back on itself (i.e., starts and ends with the same vertex) we specifically call it a **circuit** rather than a path. Thus, we will restrict the term *path* to open-ended trips and the word *circuit* to closed trips.

Here are a few examples of circuits in Fig. 8:

- *A*, *D*, *E*, *B*, *A* is a circuit of length 4. Even though it appears like the circuit designates *A* as the starting (and ending) vertex, a circuit is independent of where we designate the start. In other words, the same circuit can be written as *D, E, B, A, D* or *E, B, A, D, E*, etc. They are all the same circuit, but we have to choose one (arbitrary) vertex to start the list.
- *B, C, B* is a circuit of length 2. This is possible because of the double edge *BC*. On the other hand, *B, A, B* is not a circuit because the edge *AB* is being traveled twice. (Just as in a path, the edges of a circuit have to be distinct.)
- *E, E* is a circuit of length 1. A loop is the only way to have a circuit of length 1.

In Example 9 we saw several examples of paths (and circuits) that are part of the graph in Fig. 8, but the important idea we will discuss next in this: Can the path (or circuit) be the entire graph, not just a part of it? In other words, we want to consider the possibility of a path (or a circuit) that *exhausts* all the edges of the graph.

An **Euler path** (named after Leonhard Euler) is a path that covers *all* the edges of the graph. Likewise, an **Euler circuit** is a circuit that covers all the edges of the graph. In other words, we have an Euler path (or circuit) when the entire graph can be written as a path (or circuit).

EXAMPLE 10 EULER PATHS AND EULER CIRCUITS

Figures 9, 10, and 11 illustrate the three possibilities that can occur:

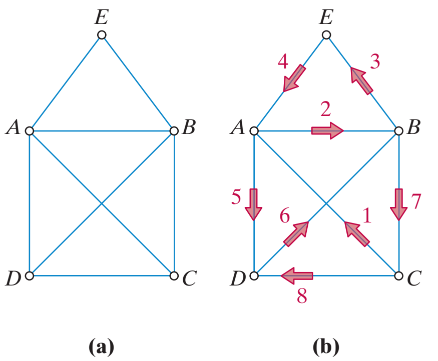


FIGURE 9 An Euler path starting at *C* and ending at *D*.

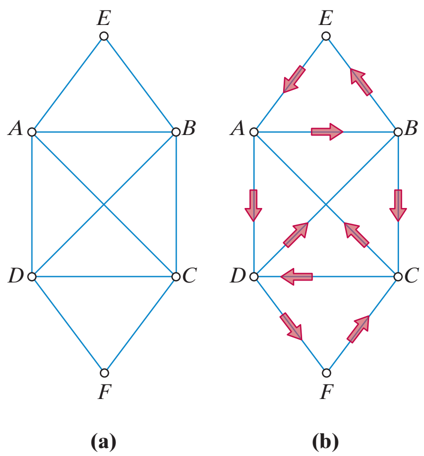


FIGURE 10 An Euler circuit.

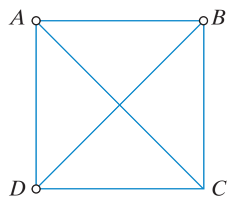


FIGURE 11 No Euler path or circuit.

- The graph in Fig. 9(a) has an Euler path—in fact, it has several. One of the possible Euler paths is shown in Fig. 9(b). The path starts at C and ends at D—just follow the arrows and you will be able to "trace" the edges of the graph without retracing any (just like in elementary school).
- The graph in Fig. 10(a) has many possible Euler circuits. One of them is shown in Fig. 10(b). Just follow the arrows. Unlike the Euler path in Fig. 9(b), the arrows are not numbered. You can start this circuit at any vertex of the graph, follow the arrows, and you will return to the starting vertex having covered all the edges once.
- The graph in Fig. 11 has neither an Euler path nor an Euler circuit. That's the way it goes sometimes—some graphs just don't have it!

We introduced the idea of a *connected* or *disconnected* graph in Example 6. Formally, we say that a graph is **connected** if you can get from any vertex to any other vertex along some path of the graph. Informally, this says that you can get from any point to any other point by "hiking" along the edges of the graph. Even more informally, it means that the graph is made of one "piece." A graph that is not connected is called **disconnected** and consists of at least two (maybe more) separate "pieces" we call the **components** of the graph.

EXAMPLE 11 BRIDGES

Figure 12 shows three different graphs. The graph in Fig. 12(a) is connected; the graph in Fig. 12(b) is disconnected and has two components; the graph in Fig. 12(c) is disconnected and has three components (the isolated vertex *G* is a component—that's as small a component as you can get!).

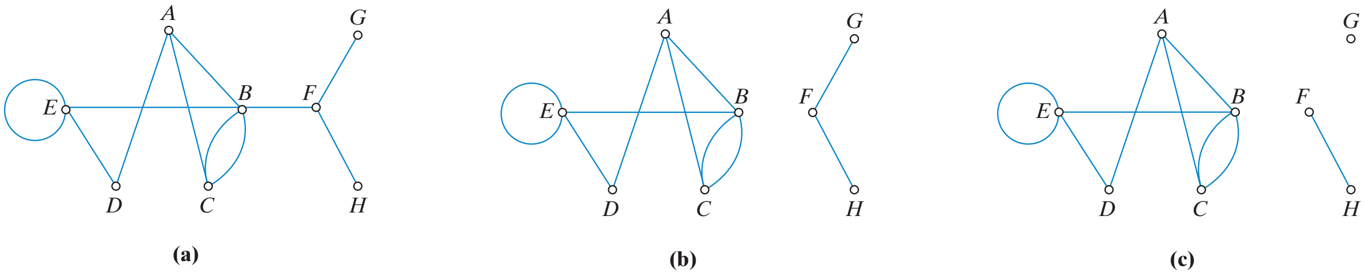


FIGURE 12 (a) A connected graph, (b) two components, and (c) three components.

Notice that the only difference between the *disconnected* graph in Fig. 12(b) and the *connected* graph in Fig. 12(a) is the edge BF. Think of *BF* as a "bridge" that connects the two components of the graph in Fig. 12(b). Not surprisingly, we call such an edge a *bridge*. A **bridge** in a connected graph is an edge that keeps the graph connected—if the bridge were not there, the graph would be disconnected. The graph in Fig. 12(a) has three bridges: *BF*, *FG*, and *FH*.

For the reader's convenience, Table 1 shows a summary of the basic graph concepts we have seen so far.

Vertices

- **adjacent:** any two vertices connected by an edge
- **vertex set:** the set of vertices in a graph
- **degree:** number of edges meeting at the vertex
- **odd (even):** degree is an odd (even) number
- **isolated:** no edges connecting the vertex (i.e., degree is 0)

Edges

- **adjacent:** two edges that share a vertex
- **loop:** an edge that connects a vertex with itself
- **multiple edges:** more than one edge connecting the same two vertices
- **edge list:** a list of all the edges in a graph
- **bridge:** an edge in a connected graph without which the graph would be disconnected

Paths and circuits

- **path:** a sequence of edges each adjacent to the next, with no edge included more than once, and starting and ending at different vertices
- **circuit:** same as a path, but starting and ending at the same vertex
- **Euler path:** a path that covers all the edges of the graph
- **Euler circuit:** a circuit that covers all the edges of the graph
- **length:** number of edges in a path or a circuit

Graphs

- **simple:** a graph with no loops or multiple edges
- **connected:** there is a path going from any vertex to any other vertex
- **disconnected:** not connected; consisting of two or more components
- **clique:** a set of completely interconnected vertices in the graph (every vertex is connected to every other vertex by an edge)

TABLE 1 Glossary of basic graph concepts