GLOBAL EDITION

Signals, Systems, and Transforms

FIFTH EDITION

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SIGNALS, SYSTEMS, AND TRANSFORMS

FIFTH EDITION

Convolution

The convolution property states that if

$$f_1(t) \overset{\mathscr{F}}{\longleftrightarrow} F_1(\omega) \quad \text{and} \quad f_2 \overset{\mathscr{F}}{\longleftrightarrow} F_2(\omega),$$

then convolution of the time-domain waveforms has the effect of multiplying their frequency-domain counterparts. Thus,

$$f_1(t)*f_2(t) \overset{\mathscr{F}}{\longleftrightarrow} F_1(\omega)F_2(\omega), \tag{5.17}$$

where

$$f_1(t)*f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau = \int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)d\tau.$$

Also, by applying the duality property to (5.17), it is shown that multiplication of time-domain waveforms has the effect of convolving their frequency-domain representations. This is sometimes called the *multiplication property,*

$$f_1(t)f_2(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{2\pi}F_1(\omega)*F_2(\omega), \tag{5.18}$$

where

$$F_1(\omega)*F_2(\omega) = \int_{-\infty}^{\infty} F_1(\lambda)F_2(\omega-\lambda)d\lambda = \int_{-\infty}^{\infty} F_1(\omega-\lambda)F_2(\lambda)d\lambda.$$

Engineers make frequent use of the convolution property in analyzing the interaction of signals and systems.

EXAMPLE 5.10 **The time-convolution property of the Fourier transform**

Chapter 3 discusses the response of linear time-invariant systems to input signals. A block diagram of a linear system is shown in Figure 5.9(a). If the output of the system in response to an impulse function at the input is described as $h(t)$, then $h(t)$ is called the *impulse response* of the system. The output of the system in response to any input signal can then be determined by convolution of the impulse response, $h(t)$, and the input signal, $x(t)$:

$$y(t) = x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

Using the convolution property of the Fourier transform, we can find the frequency spectrum of the output signal from

$$Y(\omega) = X(\omega)H(\omega),$$

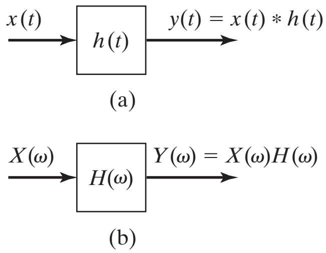

Figure 5.9 A linear time-invariant system.

where

$$h(t) \overset{\mathscr{F}}{\longleftrightarrow} H(\omega), x(t) \overset{\mathscr{F}}{\longleftrightarrow} X(\omega), \quad \text{and} \quad y(t) \overset{\mathscr{F}}{\longleftrightarrow} Y(\omega).$$

The function $H(\omega)$ is the system transfer function discussed in Section 4.5. A block diagram of the signal/system relationship in the frequency domain is shown in Figure 5.9(b). ■

The application described in Example 5.10 and other applications of the convolution property are explored more fully in Chapter 6.

Frequency Shifting

The frequency shifting property is stated mathematically as

$$x(t)e^{j\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} X(\omega - \omega_0). \tag{5.19}$$

This property was demonstrated in the derivation of (5.9), without our having recognized it.

Also, by applying the duality property to (5.17), it is shown that multiplication of time-domain waveforms has the effect of convolving their frequency-domain representations. This is sometimes called the *modulation property* and sometimes called the *multiplication property*.

EXAMPLE 5.11 **The modulation property of the Fourier transform**

Consider two signals such as

$$g_1(t) = 2\cos(200\pi t) \quad \text{and} \quad g_2(t) = 5\cos(1000\pi t)$$

that are multiplied to give

$$g_3(t) = g_1(t)g_2(t) = 10\cos(200\pi t)\cos(1000\pi t).$$

The modulation (multiplication) property can be applied to find

$$G_3(\omega) = \frac{1}{2\pi} G_1(\omega)*G_2(\omega).$$

Applying (5.11), we find

$$G_1(\omega) = 2\pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)] \text{ and}$$
$$G_2(\omega) = 5\pi[\delta(\omega - 1000\pi) + \delta(\omega + 1000\pi)].$$

Then

$$\begin{aligned} G_3(\omega) &= 5\pi[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)]*[\delta(\omega - 1000\pi) + \delta(\omega + 1000\pi)] \\ &= 5\pi\begin{bmatrix} \delta(\omega - 200\pi)*\delta(\omega - 1000\pi) + \delta(\omega + 200\pi)*\delta(\omega - 1000\pi) \\ +\delta(\omega - 200\pi)*\delta(\omega + 1000\pi) + \delta(\omega + 200\pi)*\delta(\omega + 1000\pi) \end{bmatrix}. \end{aligned}$$

Each of the convolutions in the solution for $G_3(\omega)$ is of the general form

$$\delta(\omega + \omega_1)*\delta(\omega + \omega_2) = \int_{-\infty}^{\infty} \delta(\lambda + \omega_1)\delta(\lambda - \omega - \omega_2)d\lambda.$$

Because the second impulse function in the convolution integral is non-zero only when $\lambda = \omega + \omega_2$, we see that

$$\delta(\omega + \omega_1)*\delta(\omega + \omega_2) = \delta(\omega + \omega_1 + \omega_2)\int_{-\infty}^{\infty} \delta(\lambda - \omega - \omega_2)d\lambda = \delta(\omega + \omega_1 + \omega_2).$$

We can apply this result for each of the convolutions involved in computing $G_3(\omega)$ to get

$$G_3(\omega) = 5\pi[\delta(\omega - 800\pi) + \delta(\omega - 800\pi) + \delta(\omega - 1200\pi) + \delta(\omega - 1200\pi)].$$

Referring again to (5.11) we see that

$$g_3(t) = 5\cos(800\pi t) + 5\cos(1200\pi t).$$ ■

EXAMPLE 5.12 **The frequency-shift property of the Fourier transform**

In the generation of communication signals, often two signals such as

$$g_1(t) = 2\cos(200\pi t) \quad \text{and} \quad g_2(t) = 5\cos(1000\pi t)$$

are multiplied together to give

$$g_3(t) = g_1(t)g_2(t) = 10\cos(200\pi t)\cos(1000\pi t).$$

We can use the frequency-shifting property to find the frequency spectrum of $g_3(t)$. We rewrite the product waveform $g_3(t)$ by using Euler's identity on the second cosine factor:

$$\begin{aligned} g_3(t) &= 10\cos(200\pi t)\,\frac{e^{j1000\pi t} + e^{-j1000\pi t}}{2} \\ &= 5\cos(200\pi t)e^{j1000\pi t} + 5\cos(200\pi t)e^{-j1000\pi t}. \end{aligned}$$

The Fourier transform of this expression is found from the properties of linearity (5.10), frequency shifting (5.19), and the transform of cos $(\omega_0 t)$ from (5.11):

$$G_3(\omega) = 5\pi[\delta(\omega - 200\pi - 1000\pi) + \delta(\omega + 200\pi - 1000\pi)] + 5\pi[\delta(\omega - 200\pi + 1000\pi) + \delta(\omega + 200\pi + 1000\pi)].$$

In final form, we write

$$G_3(\omega) = 5\pi[\delta(\omega - 1200\pi) + \delta(\omega - 800\pi) + \delta(\omega + 800\pi) + \delta(\omega + 1200\pi)].$$

The frequency spectra of $g_1(t)$, $g_2(t)$, and $g_3(t)$ are shown in Figure 5.10.

It is of interest to engineers that the inverse Fourier transform of $G_3(\omega)$ is

$$\begin{aligned} g_3(t) &= \mathscr{F}^{-1}\{5\pi[\delta(\omega - 1200\pi) + \delta(\omega + 1200\pi)]\} \\ &\quad + \mathscr{F}^{-1}\{5\pi[\delta(\omega - 800\pi) + \delta(\omega + 800\pi)]\} \\ &= 5\cos 1200\pi t + 5\cos 800\pi t. \end{aligned}$$

The product of two sinusoidal signals has produced a sum of two sinusoidal signals. (This is also seen from trigonometric identities.) One has the frequency that is the sum of the frequencies of the two original signals, whereas the other has the frequency that is the difference of the two original frequencies. This characteristic is often used in the process of generating signals for communication systems and in applications such as radar and sonar.

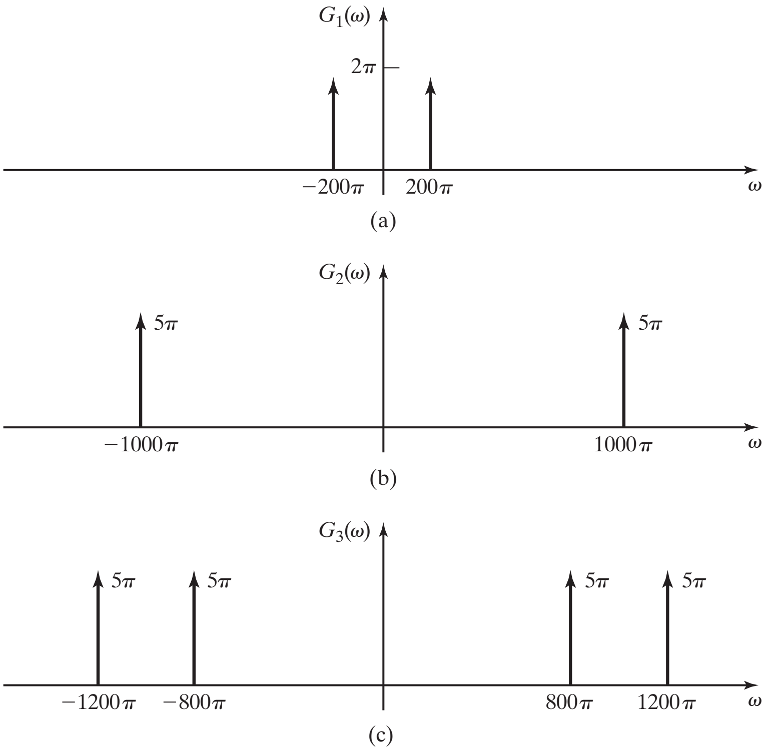

Figure 5.10 The frequency spectrum of $10\cos(200\pi t)\cos(1000\pi t)$.

These results are confirmed by the following MATLAB program:

```
% This MATLAB program finds the Fourier transform of the product
% of two sinusoidal signals using the symbolic math function
% "fourier".
%
syms t
g1=2*cos(200*pi*t)
g2=5*cos(1000*pi*t)
% Multiply the two sinusoidal signals,'g3 = g1*g2='
g3=g1*g2
% Compute the Fourier transform,'G3=fourier(g3)'
G3=fourier(g3)
```

■

Time Integration

If

$$f(t) \stackrel{\mathcal{F}}{\longleftrightarrow} F(\omega),$$

then

$$g(t) = \int_{-\infty}^{t} f(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega) = G(\omega), \tag{5.20}$$

where

$$F(0) = F(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} f(t)dt,$$

from (5.1). If $f(t)$ has a nonzero time-averaged value (dc value), then $F(0) \neq 0$.

The time-integration property of the Fourier transform will now be proved. Consider the convolution of a generic waveform $f(t)$ with a unit step function:

$$f(t)*u(t) = \int_{-\infty}^{\infty} f(\tau)u(t-\tau)d\tau.$$

The unit step function $u(t-\tau)$ has a value of zero for $t < \tau$ and a value of 1 for $t > \tau$. This can be restated as

$$u(t-\tau) = \begin{cases} 1, & \tau < t \\ 0, & \tau > t \end{cases},$$

and, therefore,

$$f(t)*u(t) = \int_{-\infty}^{t} f(\tau)d\tau. \tag{5.21}$$

The integration property yields

$$\int_{-\infty}^{t} f(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega).$$

The factor $F(0)$ in the second term on the right follows from the sifting property (2.42) of the impulse function.

EXAMPLE 5.13 **The time-integration property of the Fourier transform**

Figure 5.1l(a) shows a linear system that consists of an integrator. As discussed in Section 1.2, this can be physically realized electronically by a combination of an operational amplifier, resistors, and capacitors. The input signal is a pair of rectangular pulses as shown in Figure 5.11(b). Using time-domain integration we can see that the output signal would be a triangular waveform, as shown in Figure 5.11(c). We wish to know the frequency spectrum of the output signal. We have not derived the Fourier transform of a triangular wave; however, we do know the Fourier transform of a rectangular pulse such as is present at the input of the system. Using the properties of linearity and time shifting, we can write the input signal as

$$x(t) = A\ \mathrm{rect}\left[\frac{t + t_1/2}{t_1}\right] - A\,\mathrm{rect}\left[\frac{t - t_1/2}{t_1}\right]$$

and

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The Fourier transform of $x(t)$ is

$$\begin{aligned} X(\omega) &= At_1\,\mathrm{sinc}(t_1\omega/2)[e^{j\omega t_1/2} - e^{-j\omega t_1/2}] \\ &= 2jAt_1\,\mathrm{sinc}(t_1\omega/2)\ \sin(t_1\omega/2) \\ &= j\omega At_1^2\,\mathrm{sinc}(t_1\omega/2)\left[\frac{\sin(t_1\omega/2)}{t_1\omega/2}\right] \\ &= j\omega At_1^2\,\mathrm{sinc}^2(t_1\omega/2). \end{aligned}$$

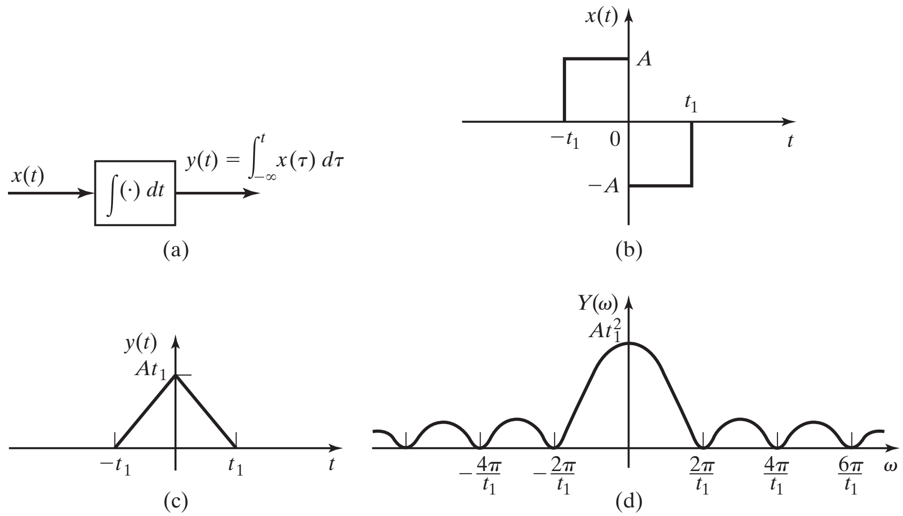

Figure 5.11 System and waveforms for Example 5.13.