Quantitative Methods

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Quantitative Methods

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Test yourself

Q3. Calculate the present value of a continuous revenue stream for 5 years, at a constant rate of £2,000 a year, if the discount rate is 5%.

Further rules of differentiation

Key definition

The **chain rule** states that if y is a function of u, which is itself a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule shows how to differentiate a function of a function (known as a composite function): differentiate the outer function and multiply by the derivative of the inner function.

The formal approach to differentiating the function $y = (3x^2 + 4)^7$ is to let $u = 3x^2 + 4$ so that $y = u^7$.

Hence

$$\frac{dy}{du} = 7u^6 = 7(3x^2 + 4)^6.$$

and

$$\frac{du}{dx} = 6x.$$

The chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 7(3x^2 + 4)^6 \times 6x = 42x(3x^2 + 4)^6.$$

With practice, the logic can be done in your head and the answer written down without working. For example, to differentiate $y = (x^2 + 3x - 9)^4$, you first differentiate the outer power function to get $4(x^2 + 3x - 9)^3$, and then multiply by the derivative of the inner function, $x^2 + 3x + 9$, which is $2x + 3$, so

$$\frac{dy}{dx} = 4(x^2 + 3x - 9)^3\,(2x + 3).$$

Key definition

The **product rule** states that if $y = uv$, where u and v are both functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The product rule shows how to differentiate the product of two functions: multiply each function by the derivative of the other and add.

To differentiate $y = x^2e^{3x}$, we write $u = x^2$ and $v = e^{3x}$ so that

$$\frac{du}{dx} = 2x$$

and

$$\frac{dv}{dx} = 3e^{3x}.$$

The product rule then gives

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= x^2 \times 3e^{3x} + e^{3x} \times 2x\\ &= x(3x + 2)e^{3x}.\end{aligned}$$

Key definition

The **quotient rule** states that if

$$y = \frac{u}{v}$$

where u and v are both functions of x, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

The quotient rule shows how to differentiate the quotient of two functions: bottom times derivative of top, minus top times derivative of bottom, all over bottom squared.

To differentiate

$$y = \frac{2x - 1}{3x + 1},$$

we write $u = 2x - 1$ and $v = 3x + 1$ so that

$$\frac{du}{dx} = 2$$

and

$$\frac{dv}{dx} = 3.$$

The quotient rule then gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(3x + 1)(2) - (2x - 1)(3)}{(3x + 1)^2} \\ &= \frac{6x + 2 - 6x + 3}{(3x + 1)^2} \\ &= \frac{5}{(3x + 1)^2}\end{aligned}$$

Test yourself

Q4. Differentiate:

(a) $\ln(x^2 + 4)$

(b) $x\sqrt{2x + 1}$

(c) $\dfrac{x}{3x + 4}$

Worked example

A firm's demand function is given by $P = \sqrt{100 - 2Q}$.

(a) Find expressions for total revenue and marginal revenue.

(b) Find the value of Q that maximises total revenue.

Solution

(a) $TR = PQ = Q\sqrt{100 - 2Q}$.

An expression for marginal revenue can be obtained by differentiating *TR* using the product rule. If we write

$$u = Q;\ v = (100 - 2Q)^{\frac{1}{2}}$$

then

$$\frac{du}{dQ} = 1;\ \frac{dv}{dQ} = \frac{1}{2}(100 - 2Q)^{-\frac{1}{2}}(-2) = \frac{-1}{\sqrt{100 - 2Q}}\ \text{(chain rule).}$$

The product rule gives

$$\begin{aligned} MR &= -\frac{Q}{\sqrt{100 - 2Q}} + 1 \times \sqrt{100 - 2Q} \\ &= \frac{-Q + (100 - 2Q)}{\sqrt{100 - 2Q}} \\ &= \frac{100 - 3Q}{\sqrt{100 - 2Q}} \end{aligned}$$

(b) Total revenue is maximised when

$$\frac{d(TR)}{dQ} = 0.$$

From part (a) this occurs when $100 - 3Q = 0$, so $Q = 100 / 3$.

To classify this stationary point we differentiate a second time by using the quotient rule to find an expression for the derivative of *MR*. If we write $u = 100 - 3Q$; $v = (100 - 2Q)^{\frac{1}{2}}$, then

$$\frac{du}{dQ} = -3;\ \frac{dv}{dQ} = \frac{-1}{\sqrt{100 - 2Q}}$$

where the chain rule has been used to differentiate *v*, as before.

The quotient rule gives

$$\frac{d^2(TR)}{dQ^2} = \frac{\sqrt{100 - 2Q} \times (-3) - (100 - 3Q) \times \dfrac{-1}{\sqrt{100 - 2Q}}}{100 - 2Q}$$

There is little to be gained by simplifying this expression because we are only interested in its sign when $Q = 100/3$. If this is substituted, then the second term in the numerator is zero and we are left with

$$\frac{-3\sqrt{\frac{100}{3}}}{\frac{100}{3}} = -\frac{3\sqrt{3}}{10}$$

This is negative, confirming that the stationary point is a maximum.

Test yourself

Q5. A firm's total cost function is $TC = 20\,e^{0.01Q}$. Find the value of Q that minimises the average cost, and verify that at this level of output $AC = MC$.

The result of this "Test yourself" question is true for all total cost functions and can be proved in general easily. Average cost is defined to be total cost divided by quantity so

$$AC = \frac{TC}{Q}.$$

If we put $u = TC$ and $v = Q$, then

$$\frac{du}{dQ} = \frac{d(TC)}{dQ} = MC$$

and

$$\frac{dQ}{dQ} = 1.$$

The quotient rule gives

$$\frac{d(AC)}{dQ} = \frac{Q \times MC - TC \times 1}{Q^2}.$$

At a stationary point,

$$\frac{d(AC)}{dQ} = 0$$

so $Q \times MC = TC$.

This can be rearranged as

$$MC = \frac{TC}{Q},$$

so $MC = AC$ at a stationary point. This shows that, when drawn on the same diagram, the graphs of MC and AC intersect at the stationary points of the AC curve.

Test yourself

Q6. (a) Use the quotient rule to show that if

$$\frac{d(AC)}{dQ} = \frac{Q \times MC - TC}{Q^2}$$

then

$$\frac{d^2(AC)}{dQ^2} = \frac{1}{Q}\frac{d(MC)}{dQ} - \frac{2}{Q^3}(Q \times MC - TC).$$

(b) Use the result of part (a) to show that at a stationary point on the *AC* curve,

$$\frac{d^2(AC)}{dQ^2} = \frac{1}{Q}\frac{d(MC)}{dQ}$$

and deduce that at a minimum point on the *AC* curve, the graph of *MC* is increasing.

In Chapter 3, elasticity of demand was defined to be

$$E = \frac{P}{Q} \times \frac{dQ}{dP}.$$

This formula assumes that Q is given in terms of P. Unfortunately this is not always the case. The demand function may well be given the other way round, with P written in terms of Q. It can be difficult (and sometimes impossible) to rearrange this to make Q the subject. The chain rule provides an alternative way of dealing with this. The formal statement of the chain rule is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

so that

$$\frac{dQ}{dQ} = \frac{dQ}{dP} \times \frac{dP}{dQ}.$$

Of course, $dQ/dQ = 1$. Hence

$$\frac{dQ}{dP} = \frac{1}{dP/dQ}$$

so an alternative formula for elasticity is

$$\boxed{E = \frac{P}{Q} \times \frac{1}{dP/dQ}}$$