
# Electrical Engineering

## CONCEPTS AND APPLICATIONS

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# Electrical Engineering

## *Concepts and Applications*

$$= \frac{V_s}{s} + \frac{\omega_0^2 V_s/(j2\omega_n(-\alpha + j\omega_n))}{s + \alpha - j\omega_n} - \frac{\omega_0^2 V_s/(j2\omega_n(-\alpha - j\omega_n))}{s + \alpha + j\omega_n}$$

$$= \frac{V_s}{s} - \frac{\omega_0^2 V_s(s + 2\alpha)/(\alpha^2 + \omega_n^2)}{s^2 + 2\alpha s + \alpha^2 + \omega_n^2},\ \omega_0^2 = \alpha^2 + \omega_n^2$$

$$= \frac{V_s}{s} - \frac{V_s(s + 2\alpha)}{(s + \alpha)^2 + \omega_n^2}$$

$$= V_s\left(\frac{1}{s} - \frac{s + \alpha}{(s + \alpha)^2 + \omega_n^2} - \frac{\alpha}{(s + \alpha)^2 + \omega_n^2}\right)$$

$$= V_s\left(\frac{1}{s} - \frac{s + \alpha}{(s + \alpha)^2 + \omega_n^2} - \frac{\alpha}{\omega_n}\frac{\omega_n}{(s + \alpha)^2 + \omega_n^2}\right) \quad \textbf{(5.63)}$$

Now, the inverse Laplace transform of Equation (5.63) can be easily obtained using Table B.1 (see Appendix B) as follows:

$$v_C(t) = V_s - V_s\cos(\omega_n t)e^{-\alpha t} - \frac{\alpha V_s}{\omega_n}\sin(\omega_n t)e^{-\alpha t} \quad \textbf{(5.64)}$$

Similar to the previous two cases, the last two terms in Equation (5.64) decay to zero as $t$ tends to infinity because $\alpha$ is always positive. In addition, the first term is the steady-state capacitor voltage. The circuit in this case is said to be **underdamped**. The current through the inductor can be calculated by differentiating the capacitor voltage and multiplying by the capacitance:

$$i_L(t) = i_C(t) = C\frac{dv_C(t)}{dt} \quad \textbf{(5.65)}$$

## EXAMPLE 5.15 Series RLC Circuit

The DC voltage source of the circuit shown in Figure 5.38 is connected to the series RLC circuit by closing the switch at $t = 0$. The initial conditions are: $i_L(0) = 0$ and $v_C(0) = 0$. Find the voltage and the current across the capacitor if:

**a.** $R = 60\ \Omega$
**b.** $R = 40\ \Omega$
**c.** $R = 30\ \Omega$

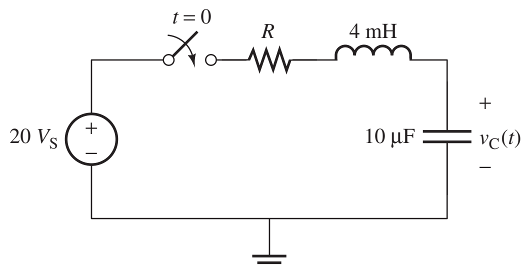


**FIGURE 5.38** Circuit for Example 5.15.

**SOLUTION**

To find the circuit's conditions, calculate parameters $\alpha$, $\omega_0$, and $\zeta$:

**a.** In this case:

$$\alpha = \frac{R}{2L} = \frac{60}{2 \times 4 \times 10^{-3}} = 7500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times 10^{-3} \times 10 \times 10^{-6}}} = 5000$$

$$\zeta = \frac{\alpha}{\omega_0} = \frac{7500}{5000} = 1.5$$

Because $\zeta > 1$, the circuit is overdamped and the roots of the characteristic function can be calculated using Equations (5.54) and (5.55) as follows:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1.91 \times 10^3$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -1.31 \times 10^4$$

Substituting these values in Equation (5.53):

$$V_C(s) = \frac{20 \times 25 \times 10^6}{s\left(s^2 + 15 \times 10^3 s + 25 \times 10^6\right)} = \frac{5 \times 10^8}{s(s + 1.91 \times 10^3)(s + 1.31 \times 10^4)}$$

$$V_C(s) = \frac{20}{s} - \frac{23.42}{s + 1.91 \times 10^3} + \frac{3.42}{s + 1.31 \times 10^4}$$

The corresponding inverse Laplace transform is:

$$v_C(t) = 20 - 23.42e^{-1.91 \times 10^3 t} + 3.42e^{-1.31 \times 10^4 t}$$

The current flowing through the circuit is given by:

$$i_C(t) = C\frac{dv_C(t)}{dt} = 0 - 10 \times 10^{-6} \times 23.42 \times -1.91 \times 10^3 e^{-1.91 \times 10^3 t} + 10 \times 10^{-6} \times 3.42 \times -1.31 \times 10^4 e^{-1.31 \times 10^4 t}$$

$$i_C(t) = 0.447e^{-1.91 \times 10^3 t} - 0.447e^{-1.31 \times 10^4 t}$$

Plots of capacitor voltage and current are shown in Figure 5.39.

**b.** When the resistor changes to 40 Ω:

$$\alpha = \frac{R}{2L} = \frac{40}{2 \times 4 \times 10^{-3}} = 5000$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times 10^{-3} \times 10 \times 10^{-6}}} = 5000$$

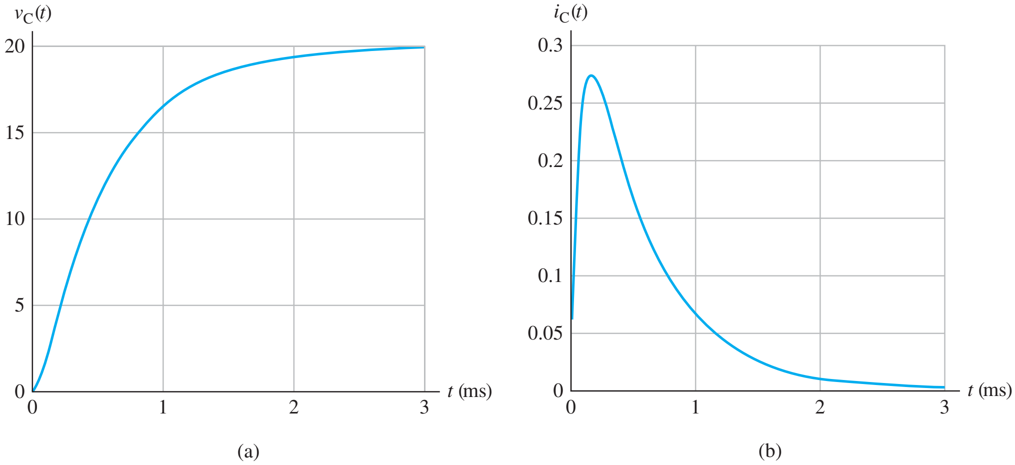


**FIGURE 5.39** $R = 60\ \Omega$: (a) the capacitor voltage; (b) the current through the circuit.

*(continued)*

**EXAMPLE 5.15** **Continued**

$$\zeta = \frac{\alpha}{\omega_0} = \frac{5000}{5000} = 1$$

Because $\zeta = 1$, the circuit is critically damped and the roots of the characteristic function are equal to $-\alpha$. Therefore:

$$s_1 = s_2 = -5 \times 10^3$$

Substituting these values in Equation (5.58):

$$V_C(s) = \frac{20 \times 25 \times 10^6}{s(s + 5 \times 10^3)^2} = \frac{V_s}{s} + \frac{k_1}{(s + 5 \times 10^3)} + \frac{k_2}{(s + 5 \times 10^3)^2}$$

The residues $k_1$ and $k_2$ are computed as follows (see Appendix B):

$$k_2 = (s + 5 \times 10^3)^2\, Y(s)\big|_{s=-5\times10^3} = \frac{20 \times 25 \times 10^6}{-5 \times 10^3} = \frac{5 \times 10^8}{-5 \times 10^3} = -10^5$$

$$k_1 = \frac{d}{ds}\left[(s + 5 \times 10^3)^2 Y(s)\right]\big|_{s=-5\times10^3} = -\frac{20 \times 25 \times 10^6}{(-5 \times 10^3)^2} = -20$$

As a result:

$$V_C(s) = \frac{20}{s} - \frac{20}{(s + 5 \times 10^3)} - \frac{10^5}{(s + 5 \times 10^3)^2}$$

The inverse Laplace transform is:

$$v_C(t) = 20 - 20e^{-5\times10^3 t} - 10^5 te^{-5\times10^3 t}$$

$$i(t) = C\frac{dv_C(t)}{dt} = 0 - 10 \times 10^{-6} \times 20 \times -5 \times 10^3 e^{-5\times10^3 t} - 10 \times 10^{-6} \times 10^5 e^{-5\times10^3 t}$$

$$-10 \times 10^{-6} \times 10^5 \times -5 \times 10^3 te^{-5\times10^3 t}$$

$$i(t) = e^{-5\times10^3 t} - e^{-5\times10^3 t} + 5 \times 10^3 te^{-5\times10^3 t} = 5 \times 10^3 te^{-5\times10^3 t}$$

Plots of capacitor voltage and current are shown in Figure 5.40.

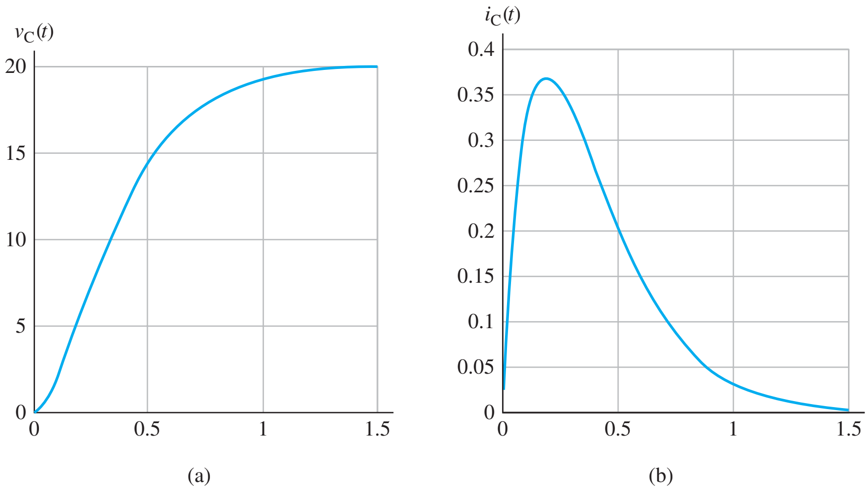


**FIGURE 5.40** *R* = 40 Ω: (a) the capacitor voltage; (b) the current through the circuit.

**c.** Finally, for $R = 30\ \Omega$:

$$\alpha = \frac{R}{2L} = \frac{30}{2 \times 4 \times 10^{-3}} = 3750$$
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times 10^{-3} \times 10 \times 10^{-6}}} = 5000$$
$$\zeta = \frac{\alpha}{\omega_0} = \frac{3750}{5000} = 0.75$$

Because $\zeta < 1$, the circuit is underdamped. The natural frequency is calculated using Equation (5.62), which is:

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} = 3307$$

The roots of the characteristic function can be calculated using Equations (5.60) and (5.61).

$$s_1 = -\alpha + j\omega_n = -3750 + j3307$$
$$s_2 = -\alpha - j\omega_n = -3750 - j3307$$

This is an underdamped case, using Equation (5.63) results in:

$$V_C(s) = \frac{5 \times 10^8}{s(s + 3750 + j3307)(s + 3750 - j3307)}$$

and

$$V_C(s) = \frac{20}{s} - \frac{20(s + 3750)}{(s + 3750)^2 + 3307^2} - \frac{20 \times 3750}{3307}\,\frac{3307}{(s + 3750)^2 + 3307^2}$$

The corresponding inverse Laplace transform is:

$$v_C(t) = 20 - 20e^{-3750t}\cos(3307t) - 22.68e^{-3750t}\sin(3307t)$$

The corresponding current that flows through the circuit is given by:

$$i_C(t) = 0.75e^{-3750t}\cos(3307t) + 0.66e^{-3750t}\sin(3307t) + 0.85e^{-3750t}\sin(3307t)$$
$$-0.75e^{-3750t}\cos(3307t)$$
$$i_C(t) = 1.51e^{-3750t}\sin(3307t)$$

Plots of capacitor voltage and current are shown in Figure 5.41.

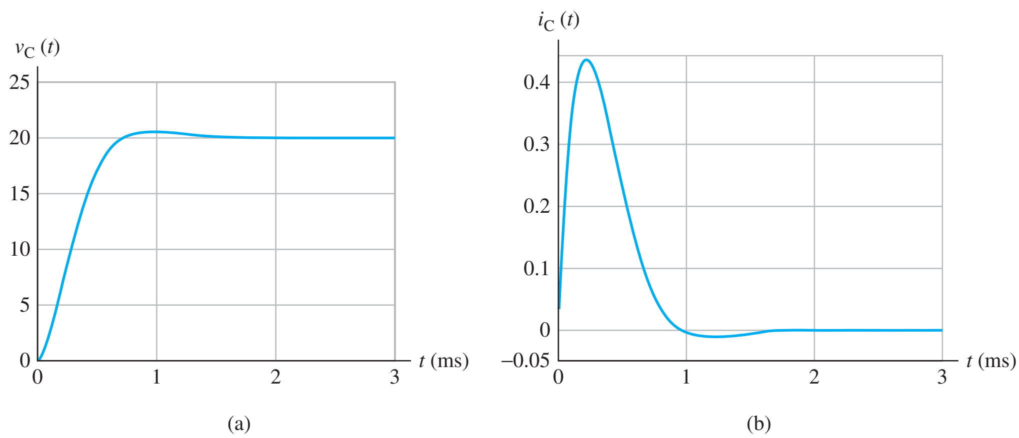


**FIGURE 5.41** $R = 30\ \Omega$: (a) The capacitor voltage; (b) the current through the circuit.

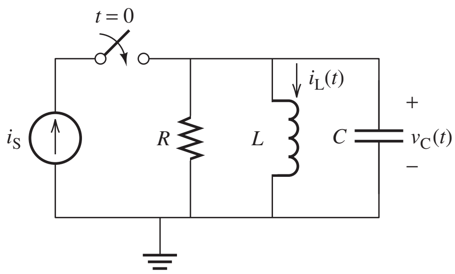


**FIGURE 5.42** A parallel RLC circuit with a DC source.

### 5.5.2 Parallel RLC Circuits with a DC Voltage Source

This section examines circuits that contain parallel resistors, inductors, and capacitors. The previous section outlined how to compute $v_C(t)$, and how to easily find the inductor current by differentiating the voltage across the capacitor. This section describes how to compute the inductor current and then calculate capacitor voltage using differentiation.

Consider the circuit shown in Figure 5.42, in which the DC source is connected to the RLC circuit by closing the switch at time $t = 0$. To find the inductor current, write KCL to find the differential equation as follows:

$$i_R(t) + i_L(t) + i_C(t) = I_s \tag{5.66}$$

In this example:

$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_L(t)}{R} = \frac{L}{R}\frac{di_L(t)}{dt} \tag{5.67}$$

$$i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dv_L(t)}{dt} = LC\frac{d^2 i_L(t)}{dt^2} \tag{5.68}$$

Plugging Equations (5.67) and (5.68) into Equation (5.66) leads to:

$$\frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) + LC\frac{d^2 i_L(t)}{dt^2} = i_s \tag{5.69}$$

Dividing both sides of Equation (5.69) by $LC$ and rearranging terms results in:

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{i_s}{LC} \tag{5.70}$$

Laplace transform of Equation (5.70) yields:

$$s^2 I_L(s) - s i_L(0) - i'_L(0) + \frac{1}{RC}\left(sI_L(s) - i_L(0)\right) + \frac{1}{LC}I_L(s) = \frac{I_s/LC}{s} \tag{5.71}$$

Setting all initial conditions to zero, the inductor current will correspond to:

$$I_L(s) = \frac{I_s/LC}{s\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)} \tag{5.72}$$

Next, consider:

The damping coefficient is:

$$\alpha = \frac{1}{2RC} \tag{5.73}$$

The undamped resonant frequency is:

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{5.74}$$

The damping ratio is:

$$\zeta = \frac{\alpha}{\omega_0} \tag{5.75}$$

Plugging these values into Equation (5.72) results in:

$$I_L(s) = \frac{I_s \omega_0^2}{s(s^2 + 2\alpha s + \omega_0^2)} \quad \textbf{(5.76)}$$

Comparing Equation (5.76) with (5.53), it is clear that the solution will be the same. The only difference between series RLC and parallel RLC circuits is the value of the damping coefficient, $\alpha$. Considering the inductor's current instead of the capacitor's voltage results in the same three cases as in the previous section, and thus $i_L(t)$ can be found.

## EXAMPLE 5.16 Parallel RLC Circuit

The DC source of the circuit shown in Figure 5.43 is connected to the parallel RLC circuit by closing the switch at $t = 0$. The initial conditions are: $i_L(0) = 0$ and $v_C(0) = 0$. Find the inductor current and the voltage across the capacitor.

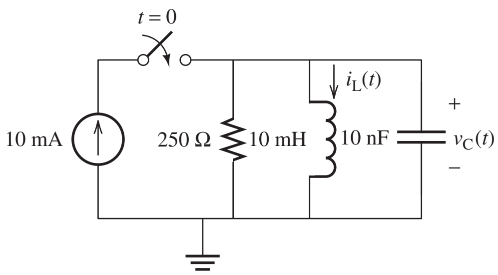


**FIGURE 5.43** Circuit for Example 5.16.

**SOLUTION**

First, compute the key parameters using Equations (5.73) through (5.75):

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 250 \times 10 \times 10^{-9}} = 2 \times 10^5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \times 10^{-3} \times 10 \times 10^{-9}}} = 10^5$$

$$\zeta = \frac{\alpha}{\omega_0} = \frac{2 \times 10^5}{10^5} = 2$$

Because $\zeta > 1$, the circuit is overdamped and the roots of the characteristic function can be calculated using Equations (5.57) and (5.55) as follows:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -2.6795 \times 10^4$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -3.7321 \times 10^5$$

Substituting these values into Equation (5.76) leads to:

$$I_L(s) = \frac{10 \times 10^{-3} \times 10^{10}}{s(s^2 + 4 \times 10^5 s + 10^{10})} = \frac{10^8}{s(s + 2.6795 \times 10^4)(s + 3.7321 \times 10^5)}$$

$$I_L(s) = \frac{10 \times 10^{-3}}{s} - \frac{10.7735 \times 10^{-3}}{s + 2.6795 \times 10^4} + \frac{0.7735 \times 10^{-3}}{s + 3.7321 \times 10^5}$$

The inverse Laplace transform corresponds to:

$$i_L(t) = 10 \times 10^{-3} - 10.7735 \times 10^{-3} e^{-2.6795 \times 10^4 t} + 0.7735 \times 10^{-3} e^{-3.7321 \times 10^5 t}$$

$$v_C(t) = 2.8868 e^{-2.6795 \times 10^4 t} - 2.8868 e^{-3.7321 \times 10^5 t}$$