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Thinking Mathematically

Second Edition

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systematic and take sets of two, then three, then four consecutive numbers and find the sums. For the moment, choose the first alternative.

Attack

1 = 0 + 1 Is 0 allowed?
No, numbers have to be positive.
2 = ? Can't be done.
3 = 1 + 2
4 = ? Can't be done.

Conjecture 1:
Even numbers are not the sum of consecutive numbers.

Continue the attack with more specializing:

5 = 2 + 3
6 = 1 + 2 + 3

Conjecture 1 is disproved. Continue specializing:

7 = 3 + 4
8 = ? Can't be done.

Conjecture 2:
Powers of two are not the sum of consecutive numbers.

The evidence for Conjecture 2 is rather thin, though since $1 = 2^0$ it copes nicely with 1. That is nice, because I had forgotten him. Now I predict that $16 = 2^4$ will be in trouble. More specializing is indicated, up to 16 at least. While accumulating all this data, there are a lot of other patterns emerging, concerned with sums of two, three and four numbers. These patterns should be written down as AHA!s or conjectures, even if at this stage they are not checked thoroughly. Some of them may contain important observations which will be useful as the resolution progresses.

DO THIS NOW, IF YOU HAVE NOT ALREADY

REFLECT: Pause now and notice how the conjecturing process is already well under way. Conjecturing is arising automatically by carrying out the familiar processes of specializing and generalizing. Specializing gives a feel for what is going on; detecting some underlying pattern (generalizing) and articulating it produces a conjecture which can then be examined, challenged and modified. In this case, more specializing has supported Conjecture 2.

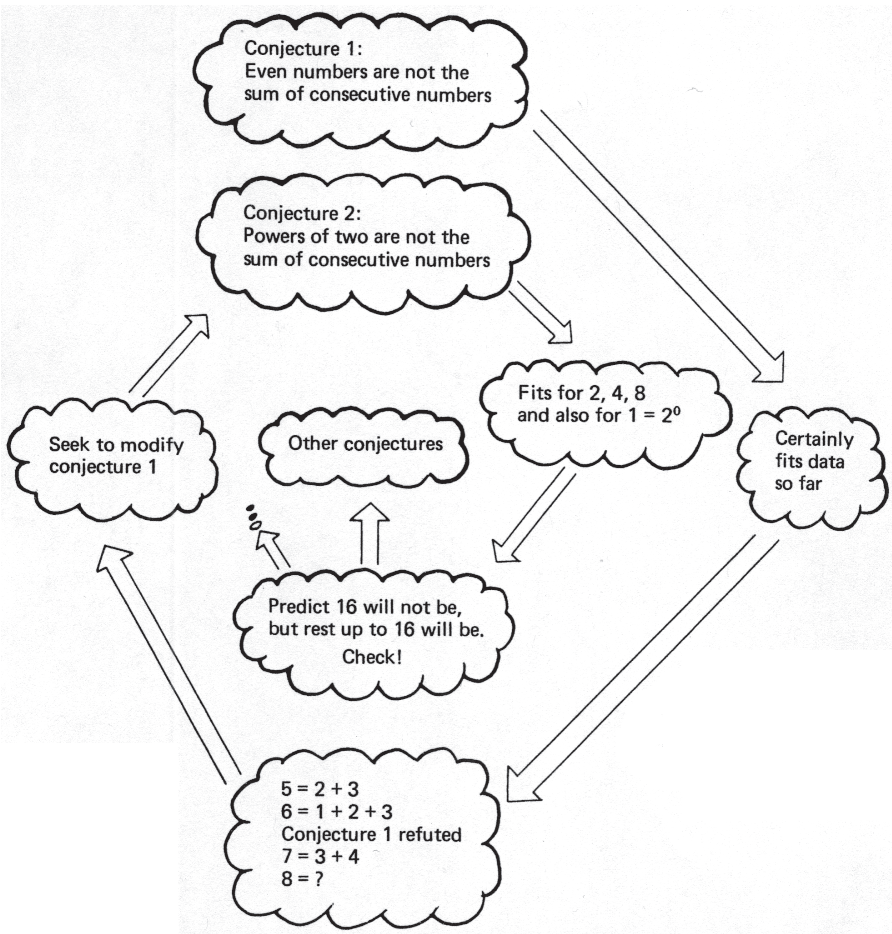
The conjecturing process so far looks something like the diagram on the next page. Conjecture 2 has been round a full cycle, and further examples seem to confirm it. Before thinking in more general terms about why it might or might not be true, notice that it can be substantially improved by articulating a second feature to which the specializing has pointed.

Conjecture 3:

(i) Powers of two cannot be expressed as sums of consecutive positive numbers.
(ii) All other numbers are sums of consecutive positive numbers.

To see whether Conjecture 2 is true for all numbers, we really have two subsidiary questions to answer:

1 How can any number that is **not** a power of two actually be written as the sum of consecutive numbers?
2 Why can a power of two **not** be expressed as a sum of consecutive numbers?



These two questions lead to a fundamental issue: what distinguishes a power of two from other numbers? How does this distinguishing property relate to the property of being a sum of consecutive numbers? I KNOW that a power of two has no prime factors other than two, by its very definition. All its factors other than one are therefore even numbers. For example, the factors of 16 are:

16, 8, 4, 2 and 1

which are all even except 1, whereas the factors of 22 are:

22, 11, 2 and 1

and there is an odd factor other than 1. I am not yet sure that this is relevant, but it is worth recording as a remark. It actually has a status somewhere between a conjecture and a fact, depending on how sceptical I want to be in my final resolution.

Conjecture 4:
Every number which is not a power of two has an odd factor other than 1.

I have recorded it as a conjecture because I do not want to spend time now justifying it. It feels right and, since it is written down clearly, I will check it in my Review. My mathematical experience reinforces my observation with a strong sense of its verifiability and I am confident enough to proceed without diverting my concentration from the main thread of the investigation. However, it helps to record such points so that they will be checked again later when I am more calm and less caught up in the flow of ideas.

How does the existence of an odd factor enable me to express the number as a sum of consecutive numbers? Specialize by examining numbers with odd factors, say multiples of 3 and 5.

$$
\begin{aligned}
3 &= 1 + 2 \\
6 &= 1 + 2 + 3 \\
9 &= 2 + 3 + 4 \\
12 &= 3 + 4 + 5 \\
15 &= 4 + 5 + 6 \\
5 &= 2 + 3 \\
10 &= 1 + 2 + 3 + 4 \\
15 &= 1 + 2 + 3 + 4 + 5 \\
20 &= 2 + 3 + 4 + 5 + 6 \\
25 &= 3 + 4 + 5 + 6 + 7
\end{aligned}
$$

A clear pattern (and hence a conjecture) is emerging here as larger multiples of 3 (and 5 respectively) can be obtained by taking the sum of 3 (or 5) consecutive numbers with larger starting numbers. It is usually worthwhile deliberately trying to articulate a conjecture in a situation like this because it forces you to clarify your feeling for what might be going on and provides something concrete to test. However, especially with a conjecture that arises during exploration, there is no need to be excessively formal. The important thing is to begin to capture the ideas involved. A first attempt might look like this:

Conjecture 5:
A number which has an odd factor can be written as the sum of consecutive numbers. Usually the odd factor will be the same as the number of terms.

It took five attempts to word Conjecture 5 so that it made sense! I have deliberately inserted the 'usually' because I am not ready to be precise, but I do not want to be put off by those mavericks at the start. In order to do anything with Conjecture 5 I am going to have to manipulate some odd factors. It would be best to INTRODUCE some symbols. Notice that odd numbers are characterized by the shape $2K + 1$, where K is an integer.

Conjecture 5A:
A number N which has an odd factor $2K + 1$ is usually the sum of $2K + 1$ consecutive numbers.

This fits all the data collected so far. The sensible thing is to try to see what is going on while checking it systematically on new examples.

Multiples of 3

$$3 \times 2 = 1 + 2 + 3$$
$$3 \times 3 = 2 + 3 + 4$$
$$3 \times 4 = 3 + 4 + 5$$
$$3 \times F = (F - 1) + F + (F + 1)$$

Multiples of 5

$$5 \times 3 = 1 + 2 + 3 + 4 + 5$$
$$5 \times 4 = 2 + 3 + 4 + 5 + 6$$
$$5 \times 5 = 3 + 4 + 5 + 6 + 7$$
$$5 \times F = (F - 2) + (F - 1) + F + (F + 1) + (F + 2)$$

Looks good! I am beginning to get the idea that I get a multiple of the middle number. Thus if

$$N = (2K + 1) \times F$$

then N is the sum of $2K + 1$ consecutive numbers, the middle one being F.

Conjecture 6:
TRY expressing N, which is $F \times (2K + 1)$, as a sum of F_3

$$
\begin{array}{lcr}
 & F & = F \\
 & (F-1) + (F+1) & = 2F \\
(F-2) & + \quad (F+2) & = 2F \\
 & \cdots & \\
(F-K) & + \quad (F+K) & = 2F
\end{array}
$$

There are $K + 1$ equations here, so the sum of all the left-hand sides is the sum of all the right-hand sides which is

$$(K \times 2F) + F$$

that is

$$(2K + 1) \times F$$

AHA! It works!

CHECK! There are $2K + 1$ consecutive terms. Oh dear! Are they all positive? Only if F is large enough. Let me see some examples of that, following the pattern of Conjecture 6.

$$
\begin{aligned}
F = 1: 3 \times 1 &= (1-1) + 1 + (1+1) \\
&= 0 + 1 + 2 \\
F = 1: 5 \times 1 &= (1-2) + (1-1) + 1 + (1+1) + (1+2) \\
&= -1 + 0 + 1 + 2 + 3 \\
F = 2: 5 \times 2 &= (2-2) + (2-1) + 2 + (2+1) + (2+2) \\
& 0 + 1 + 2 + 3 + 4 \\
F = 1: 7 \times 1 &= (1-3) + (1-2) + (1-1) + 1 + (1+1) + (1+2) + (1+3) \\
&= -2 + -1 + 0 + 1 + 2 + 3 + 4 \\
F = 2: 7 \times 2 &= (2-3) + (2-2) + (2-1) + 2 + (2+1) + (2+2) + (2+3) \\
&= -1 + 0 + 1 + 2 + 3 + 4 + 5 \\
F = 3: 7 \times 3 &= (3-3) + (3-2) + (3-1) + 3 + (3+1) + (3+2) + (3+3) \\
&= 0 + 1 + 2 + 3 + 4 + 5 + 6
\end{aligned}
$$

AHA! I can always forget the zero, and the negative terms are always counteracted by the positives. Thus

$$
\begin{aligned}
3 \times 1 &= 0 + 1 + 2 &&= 1 + 2 \\
5 \times 1 &= -1 + 0 + 1 + 2 + 3 &&= 2 + 3 \\
5 \times 2 &= 0 + 1 + 2 + 3 + 4 &&= 1 + 2 + 3 + 4 \\
7 \times 1 &= -2 + -1 + 0 + 1 + 2 + 3 + 4 &&= 3 + 4 \\
7 \times 2 &= -1 + 0 + 1 + 2 + 3 + 4 + 5 &&= 2 + 3 + 4 + 5 \\
7 \times 3 &= 0 + 1 + 2 + 3 + 4 + 5 + 6 &&= 1 + 2 + 3 + 4 + 5 + 6
\end{aligned}
$$

Perhaps I can use this to deal with the 'usually' of Conjecture 5. Recap: I KNOW that any number N with an odd factor $2K + 1$ can be written as the sum of $2K + 1$ consecutive numbers. But some of them may be negative. I WANT to show that any number N with an odd factor can be written as the sum of two or more consecutive positive numbers.

Pondering on I WANT and I KNOW for a while, I suddenly realized that I had actually finished! All I have to do is counteract the negative terms by the corresponding positive terms. There must be more positive terms than negative ones since the total sum is positive!

CHECK! What if this counteracting of positive and negative leaves me with just a single term. Oh dear! Could this happen? Specialize:

$$-2 + -1 + 0 + 1 + 2 + 3$$

has 6 terms. It is the presence of 0 which does it. To end up after the counteracting with just one term I would need to have had an even number of terms all told; but I have $2K + 1$ terms which is always odd. This idea generalizes:

Conjecture 7:

Starting with an odd number of terms including 0 the counteracting process always leaves me with an even number of consecutive positive numbers.

I am now satisfied that every number divisible by an odd factor other than 1 **can** be written as the sum of consecutive positive numbers. Flushed with success, I pause to review my work.

REVIEW: I have answered subsidiary question 1, but not question 2: why cannot a power of two be expressed as a sum of consecutive positive numbers?

Surely my work so far contains the answer? Let me see. Suppose the number N **can** be written as the sum of consecutive positive integers. Look at

$$7 = 3 + 4 \text{ and } 5 = 2 + 3$$

Now earlier I got these as

$$7 = -2 + -1 + 0 + 1 + 2 + 3 + 4$$

and

$$5 = -1 + 0 + 1 + 2 + 3$$

AHA! Why not use the counteracting idea again? Take any sum of consecutive positive integers. Extend them downwards to zero and beyond so that the negative ones counteract the additional positive ones. Now I WANT to show that if N can be written as the sum of consecutive positive numbers, then it **must** have an odd factor. AHA! It all depends whether there are an odd number of terms in the sum. I detect two cases:

(i) N has been expressed as the sum of an odd number of consecutive positive numbers.