CONCRETE MATHEMATICS

A FOUNDATION FOR COMPUTER SCIENCE

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SECOND EDITION

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5.3 TRICKS OF THE TRADE

Let's look next at three techniques that significantly amplify the methods we have already learned.

Trick 1: Going halves.

This should really be called Trick 1/2.

Many of our identities involve an arbitrary real number r. When r has the special form "integer minus one half," the binomial coefficient $\binom{r}{k}$ can be written as a quite different-looking product of binomial coefficients. This leads to a new family of identities that can be manipulated with surprising ease.

One way to see how this works is to begin with the *duplication formula*

$$r^{\underline{k}}(r-\tfrac{1}{2})^{\underline{k}} = (2r)^{\underline{2k}}/2^{2k}, \qquad \text{integer } k \geqslant 0. \tag{5.34}$$

This identity is obvious if we expand the falling powers and interleave the factors on the left side:

$$\begin{aligned} &r(r-\tfrac{1}{2})(r-1)(r-\tfrac{3}{2})\ldots(r-k+1)(r-k+\tfrac{1}{2}) \\ &\qquad\qquad = \frac{(2r)(2r-1)\ldots(2r-2k+1)}{2\cdot 2\cdot \ldots \cdot 2}. \end{aligned}$$

Now we can divide both sides by $k!^2$, and we get

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{2k}\binom{2k}{k}\Big/2^{2k}, \qquad \text{integer } k. \tag{5.35}$$

If we set $k = r = n$, where n is an integer, this yields

$$\binom{n-1/2}{n} = \binom{2n}{n}\Big/2^{2n}, \qquad \text{integer } n. \tag{5.36}$$

And negating the upper index gives yet another useful formula,

$$\binom{-1/2}{n} = \left(\frac{-1}{4}\right)^n\binom{2n}{n}, \qquad \text{integer } n. \tag{5.37}$$

For example, when $n = 4$ we have

. . . we halve . . .

$$\begin{aligned} \binom{-1/2}{4} &= \frac{(-1/2)(-3/2)(-5/2)(-7/2)}{4!} \\ &= \left(\frac{-1}{2}\right)^4 \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4} \\ &= \left(\frac{-1}{4}\right)^4 \frac{1\cdot 3\cdot 5\cdot 7\cdot 2\cdot 4\cdot 6\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 1\cdot 2\cdot 3\cdot 4} = \left(\frac{-1}{4}\right)^4\binom{8}{4}. \end{aligned}$$

Notice how we've changed a product of odd numbers into a factorial.

Identity (5.35) has an amusing corollary. Let $r = \frac{1}{2}n$, and take the sum over all integers k. The result is

$$\sum_k \binom{n}{2k}\binom{2k}{k}2^{-2k} = \sum_k \binom{n/2}{k}\binom{(n-1)/2}{k} = \binom{n-1/2}{\lfloor n/2 \rfloor}, \qquad \text{integer } n \geqslant 0 \tag{5.38}$$

by (5.23), because either $n/2$ or $(n-1)/2$ is $\lfloor n/2 \rfloor$, a nonnegative integer!

We can also use Vandermonde's convolution (5.27) to deduce that

$$\sum_k \binom{-1/2}{k}\binom{-1/2}{n-k} = \binom{-1}{n} = (-1)^n, \qquad \text{integer } n \geqslant 0.$$

Plugging in the values from (5.37) gives

$$\binom{-1/2}{k}\binom{-1/2}{n-k} = \left(\frac{-1}{4}\right)^k \binom{2k}{k}\left(\frac{-1}{4}\right)^{n-k}\binom{2(n-k)}{n-k} = \frac{(-1)^n}{4^n}\binom{2k}{k}\binom{2n-2k}{n-k};$$

this is what sums to $(-1)^n$. Hence we have a remarkable property of the "middle" elements of Pascal's triangle:

$$\sum_k \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n, \qquad \text{integer } n \geqslant 0. \tag{5.39}$$

For example, $\binom{0}{0}\binom{6}{3}+\binom{2}{1}\binom{4}{2}+\binom{4}{2}\binom{2}{1}+\binom{6}{3}\binom{0}{0} = 1\cdot 20+2\cdot 6+6\cdot 2+20\cdot 1 = 64 = 4^3$.

These illustrations of our first trick indicate that it's wise to try changing binomial coefficients of the form $\binom{2k}{k}$ into binomial coefficients of the form $\binom{n-1/2}{k}$, where n is some appropriate integer (usually 0, 1, or k); the resulting formula might be much simpler.

Trick 2: High-order differences.

We saw earlier that it's possible to evaluate partial sums of the series $\binom{n}{k}(-1)^k$, but not of the series $\binom{n}{k}$. It turns out that there are many important applications of binomial coefficients with alternating signs, $\binom{n}{k}(-1)^k$. One of the reasons for this is that such coefficients are intimately associated with the difference operator Δ defined in Section 2.6.

The difference Δf of a function f at the point x is

$$\Delta f(x) = f(x+1) - f(x);$$

if we apply Δ again, we get the second difference

$$\begin{aligned}\Delta^2 f(x) &= \Delta f(x+1) - \Delta f(x) = \big(f(x{+}2) - f(x{+}1)\big) - \big(f(x{+}1) - f(x)\big)\\ &= f(x+2) - 2f(x+1) + f(x)\,,\end{aligned}$$

which is analogous to the second derivative. Similarly, we have

$$\begin{aligned}\Delta^3 f(x) &= f(x+3) - 3f(x+2) + 3f(x+1) - f(x)\,;\\ \Delta^4 f(x) &= f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x)\,;\end{aligned}$$

and so on. Binomial coefficients enter these formulas with alternating signs.

In general, the nth difference is

$$\Delta^n f(x) = \sum_k \binom{n}{k}(-1)^{n-k} f(x+k)\,, \qquad \text{integer } n \geqslant 0. \tag{5.40}$$

This formula is easily proved by induction, but there's also a nice way to prove it directly using the elementary theory of operators. Recall that Section 2.6 defines the shift operator E by the rule

$$Ef(x) = f(x+1)\,;$$

hence the operator Δ is $E - 1$, where 1 is the identity operator defined by the rule $1f(x) = f(x)$. By the binomial theorem,

$$\Delta^n = (E-1)^n = \sum_k \binom{n}{k} E^k (-1)^{n-k}\,.$$

This is an equation whose elements are operators; it is equivalent to (5.40), since E^k is the operator that takes $f(x)$ into $f(x+k)$.

An interesting and important case arises when we consider negative falling powers. Let $f(x) = (x-1)^{\underline{-1}} = 1/x$. Then, by rule (2.45), we have $\Delta f(x) = (-1)(x-1)^{\underline{-2}}$, $\Delta^2 f(x) = (-1)(-2)(x-1)^{\underline{-3}}$, and in general

$$\Delta^n\big((x-1)^{\underline{-1}}\big) = (-1)^{\underline{n}}\,(x-1)^{\underline{-n-1}} = (-1)^n \frac{n!}{x(x+1)\ldots(x+n)}\,.$$

Equation (5.40) now tells us that

$$\begin{aligned}\sum_k \binom{n}{k}\frac{(-1)^k}{x+k} &= \frac{n!}{x(x+1)\ldots(x+n)}\\ &= x^{-1}\binom{x+n}{n}^{-1}, \qquad x \notin \{0, -1, \ldots, -n\}.\end{aligned} \tag{5.41}$$

For example,

$$\frac{1}{x} - \frac{4}{x+1} + \frac{6}{x+2} - \frac{4}{x+3} + \frac{1}{x+4} = \frac{4!}{x(x+1)(x+2)(x+3)(x+4)} = 1\Big/x\binom{x+4}{4}.$$

The sum in (5.41) is the partial fraction expansion of $n!/\bigl(x(x+1)\ldots(x+n)\bigr)$.

Significant results can be obtained from positive falling powers too. If $f(x)$ is a polynomial of degree d, the difference $\Delta f(x)$ is a polynomial of degree $d-1$; therefore $\Delta^d f(x)$ is a constant, and $\Delta^n f(x) = 0$ if $n > d$. This extremely important fact simplifies many formulas.

A closer look gives further information: Let

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x^1 + a_0 x^0$$

be any polynomial of degree d. We will see in Chapter 6 that we can express ordinary powers as sums of falling powers (for example, $x^2 = x^{\underline{2}} + x^{\underline{1}}$); hence there are coefficients b_d, b_{d-1}, ..., b_1, b_0 such that

$$f(x) = b_d x^{\underline{d}} + b_{d-1}x^{\underline{d-1}} + \cdots + b_1 x^{\underline{1}} + b_0 x^{\underline{0}}.$$

(It turns out that $b_d = a_d$ and $b_0 = a_0$, but the intervening coefficients are related in a more complicated way.) Let $c_k = k!\,b_k$ for $0 \leqslant k \leqslant d$. Then

$$f(x) = c_d\binom{x}{d} + c_{d-1}\binom{x}{d-1} + \cdots + c_1\binom{x}{1} + c_0\binom{x}{0};$$

thus, any polynomial can be represented as a sum of multiples of binomial coefficients. Such an expansion is called the *Newton series* of $f(x)$, because Isaac Newton used it extensively.

We observed earlier in this chapter that the addition formula implies

$$\Delta\left(\binom{x}{k}\right) = \binom{x}{k-1}.$$

Therefore, by induction, the nth difference of a Newton series is very simple:

$$\Delta^n f(x) = c_d\binom{x}{d-n} + c_{d-1}\binom{x}{d-1-n} + \cdots + c_1\binom{x}{1-n} + c_0\binom{x}{-n}.$$

If we now set $x = 0$, all terms $c_k\binom{x}{k-n}$ on the right side are zero, except the term with $k - n = 0$; hence

$$\Delta^n f(0) = \begin{cases} c_n, & \text{if } n \leqslant d; \\ 0, & \text{if } n > d. \end{cases}$$

The Newton series for $f(x)$ is therefore

$$f(x) \;=\; \Delta^d f(0)\binom{x}{d} + \Delta^{d-1} f(0)\binom{x}{d-1} + \cdots + \Delta f(0)\binom{x}{1} + f(0)\binom{x}{0}.$$

For example, suppose $f(x) = x^3$. It's easy to calculate

$$\begin{array}{ccccccc} f(0)=0, & & f(1)=1, & & f(2)=8, & & f(3)=27; \\ & \Delta f(0)=1, & & \Delta f(1)=7, & & \Delta f(2)=19; & \\ & & \Delta^2 f(0)=6, & & \Delta^2 f(1)=12; & & \\ & & & \Delta^3 f(0)=6. & & & \end{array}$$

So the Newton series is $x^3 = 6\binom{x}{3} + 6\binom{x}{2} + 1\binom{x}{1} + 0\binom{x}{0}$.

Our formula $\Delta^n f(0) = c_n$ can also be stated in the following way, using (5.40) with $x = 0$:

$$\sum_k \binom{n}{k}(-1)^k\left(c_0\binom{k}{0} + c_1\binom{k}{1} + c_2\binom{k}{2} + \cdots\right) \;=\; (-1)^n c_n\,, \qquad \text{integer } n \geqslant 0.$$

Here $\langle c_0, c_1, c_2, \ldots\rangle$ is an arbitrary sequence of coefficients; the infinite sum $c_0\binom{k}{0} + c_1\binom{k}{1} + c_2\binom{k}{2} + \cdots$ is actually finite for all $k \geqslant 0$, so convergence is not an issue. In particular, we can prove the important identity

$$\sum_k \binom{n}{k}(-1)^k(a_0 + a_1 k + \cdots + a_n k^n) \;=\; (-1)^n n!\, a_n\,, \qquad \text{integer } n \geqslant 0, \qquad (5.42)$$

because the polynomial $a_0 + a_1 k + \cdots + a_n k^n$ can always be written as a Newton series $c_0\binom{k}{0} + c_1\binom{k}{1} + \cdots + c_n\binom{k}{n}$ with $c_n = n!\, a_n$.

Many sums that appear to be hopeless at first glance can actually be summed almost trivially by using the idea of nth differences. For example, let's consider the identity

$$\sum_k \binom{n}{k}\binom{r-sk}{n}(-1)^k \;=\; s^n\,, \qquad \text{integer } n \geqslant 0. \qquad (5.43)$$

This looks very impressive, because it's quite different from anything we've seen so far. But it really is easy to understand, once we notice the telltale factor $\binom{n}{k}(-1)^k$ in the summand, because the function

$$f(k) \;=\; \binom{r-sk}{n} \;=\; \frac{1}{n!}(-1)^n s^n k^n + \cdots \;=\; (-1)^n s^n\binom{k}{n} + \cdots$$

is a polynomial in k of degree n, with leading coefficient $(-1)^n s^n/n!$. Therefore (5.43) is nothing more than an application of (5.42).

We have discussed Newton series under the assumption that $f(x)$ is a polynomial. But we've also seen that infinite Newton series

$$f(x) \;=\; c_0\binom{x}{0} + c_1\binom{x}{1} + c_2\binom{x}{2} + \cdots$$

make sense too, because such sums are always finite when x is a nonnegative integer. Our derivation of the formula $\Delta^n f(0) = c_n$ works in the infinite case, just as in the polynomial case; so we have the general identity

$$f(x) \;=\; f(0)\binom{x}{0} + \Delta f(0)\binom{x}{1} + \Delta^2 f(0)\binom{x}{2} + \Delta^3 f(0)\binom{x}{3} + \cdots\,, \qquad \text{integer } x \geqslant 0. \tag{5.44}$$

This formula is valid for any function $f(x)$ that is defined for nonnegative integers x. Moreover, if the right-hand side converges for other values of x, it defines a function that "interpolates" $f(x)$ in a natural way. (There are infinitely many ways to interpolate function values, so we cannot assert that (5.44) is true for all x that make the infinite series converge. For example, if we let $f(x) = \sin(\pi x)$, we have $f(x) = 0$ at all integer points, so the right-hand side of (5.44) is identically zero; but the left-hand side is nonzero at all noninteger x.)

A Newton series is finite calculus's answer to infinite calculus's Taylor series. Just as a Taylor series can be written

$$g(a+x) \;=\; \frac{g(a)}{0!}x^0 + \frac{g'(a)}{1!}x^1 + \frac{g''(a)}{2!}x^2 + \frac{g'''(a)}{3!}x^3 + \cdots\,,$$

(Since $E = 1 + \Delta$, $E^x = \sum_k \binom{x}{k}\Delta^k$; and $E^x g(a) = g(a+x)$.)

the Newton series for $f(x) = g(a+x)$ can be written

$$g(a+x) \;=\; \frac{g(a)}{0!}x^{\underline{0}} + \frac{\Delta g(a)}{1!}x^{\underline{1}} + \frac{\Delta^2 g(a)}{2!}x^{\underline{2}} + \frac{\Delta^3 g(a)}{3!}x^{\underline{3}} + \cdots\,. \tag{5.45}$$

(This is the same as (5.44), because $\Delta^n f(0) = \Delta^n g(a)$ for all $n \geqslant 0$ when $f(x) = g(a+x)$.) Both the Taylor and Newton series are finite when g is a polynomial, or when $x = 0$; in addition, the Newton series is finite when x is a positive integer. Otherwise the sums may or may not converge for particular values of x. If the Newton series converges when x is not a nonnegative integer, it might actually converge to a value that's *different* from $g(a+x)$, because the Newton series (5.45) depends only on the spaced-out function values $g(a)$, $g(a+1)$, $g(a+2)$, ...