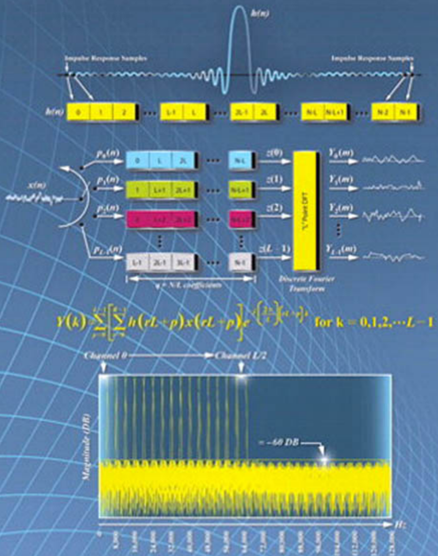
PRENTICE HALL

Practical Applications in Digital Signal Processing



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$$H(z)=\frac{Y(z)}{X(z)}=\frac{4-1.25z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$$

Now we can separate terms to get

$$Y(z)\left(1-0.75z^{-1}+0.125z^{-2}\right)=X(z)\left(4-1.25z^{-1}\right)$$

$$Y(z)=X(z)\left(4-1.25z^{-1}\right)+Y(z)\left(+0.75z^{-1}-0.125z^{-2}\right)$$

We employ the shift property of the z-transform $x(n-k)\overset{z}{\leftrightarrow}X(z)z^{-k}$ to get

$$y(n)=4x(n)-1.25x(n-1)+0.75y(n-1)-0.125y(n-2)$$

Equation 4.54

The reader should note that if we insert a discrete impulse sequence $x(n)$, composed of a unit impulse at $n=0$ and 0 at all other values, of n into the difference equation given by Equation 4.54, then we will generate the exact same sequence as given by the impulse response we derived previously, repeated here for clarity:

$$h(n)=3(0.5)^{n}+(0.25)^{n}$$

The block diagram of Equation 4.54 is easily drawn by inspection and illustrated in Figure 4.17. This is pretty powerful stuff, and as an added bonus, it's simple too. There is a lot more information that we can obtain about this system from its z-transform. For example, we can tell from the pole-zero diagram that the system is stable because all of its poles are contained within the unit circle. We also know that its ROC is within the range $0.5<|z|\leq\infty$. Since the ROC contains the unit circle, we could use the power of the z-transform to compute the frequency response of this system. We can compute the frequency response by evaluating the system transfer function on the unit circle by replacing all the occurrences of z with $e^{j\omega}$. Doing so gives us

$$H(z)\Big|_{z=e^{j\omega}}=\frac{4-1.25e^{-j\omega}}{1-0.75e^{-j\omega}+0.125e^{-j2\omega}}$$

We will pursue the frequency response topic a bit later, because right now we have other fish to fry. We still need to demonstrate the partial fraction expansion method for complex conjugate poles and higher order poles.

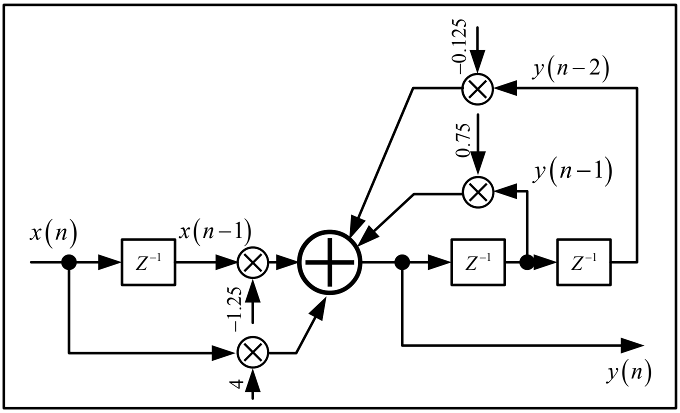

Figure 4.17 System block diagram

4.10.2.2 First Order Complex Conjugate Poles

Suppose we want to find the inverse z-transform of the following expression:

$$H(z) = \frac{z}{z^2 + 2z + 2}$$

Equation 4.55

If we divide both the numerator and the denominator by z^2, we will see that $N < M$. Therefore, by rule 2, the partial fraction expansion term $A_0 = 0$. We first factor the denominator of Equation 4.55 using the quadratic equation to get the roots of z:

$$z^2 + 2z + 2$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = (-1 \pm j)$$

The two roots are $z = -1 - j$ and $z = -1 + j$. These roots are the poles of $H(z)$. Now that we have the poles, we can expand Equation 4.55 according to rule 1 to get

$$H(z) = \sum_{k=1}^{M} \frac{A_K z}{z - p_k} = \sum_{k=1}^{2} \frac{A_K z}{z - p_k} = \frac{A_1 z}{z + 1 + j} + \frac{A_2 z}{z + 1 - j}$$

Equation 4.56

We divide both sides of the equation by z to produce

$$\frac{H(z)}{z}=\frac{A_1}{z+1+j}+\frac{A_2}{z+1-j}$$

Now we can evaluate the partial fraction coefficients for each pole in the expansion by rule 4 to get

$$A_1=(z+1+j)\frac{H(z)}{z}\bigg|_{z=(-1-j)}=\frac{(z+1+j)}{z}\left[\frac{z}{(z+1+j)(z+1-j)}\right]\bigg|_{z=(-1-j)}=\frac{1}{-2j}=\frac{j}{2}$$

$$A_2=(z+1-j)\frac{H(z)}{z}\bigg|_{z=(-1+j)}=\frac{(z+1-j)}{z}\left[\frac{z}{(z+1+j)(z+1-j)}\right]\bigg|_{z=(-1+j)}=\frac{1}{2j}=\frac{-j}{2}$$

We can complete the partial fraction expansion by substituting the coefficients $j/2$ and $-j/2$ for A_1 and A_2, respectively, in Equation 4.56 to get

$$H(z)=\frac{(j/2)z}{z+1+j}+\frac{(-j/2)z}{z+1-j}=\frac{1}{2}\left[\frac{jz}{z+1+j}+\frac{-jz}{z+1-j}\right]=\frac{j}{2}\left[\frac{z}{z+1+j}-\frac{z}{z+1-j}\right]$$

Equation 4.57

Equation 4.57 is the partial fraction expansion of $H(z)$. We can look up the inverse transform for each of the two terms in the expansion in the transform Table 4.1. Once again, from the table, we see the format of the transform pair we need:

$$k(\alpha)^n\overset{z}{\leftrightarrow}\frac{z}{z-\alpha}$$

Equation 4.58

Substituting the values in each of the two terms in Equation 4.57 into Equation 4.58, we get

$$h(n)=\frac{j}{2}(1+j)^n-\frac{j}{2}(1-j)^n$$

Equation 4.59

That's it, folks. Equation 4.59 is the inverse transform of Equation 4.57. This is indeed a valid inverse z-transform. However, it is not a satisfying result. Nobody likes imaginary operators like j in their discrete time sequence. Results like these tend to cause ulcers and other undesirable by-products of stress. Perhaps if we played with the partial fraction expansion equation a little bit, we can get it into a more palatable form. Let's begin with the original partial fraction expansion of Equation 4.57:

$$H(z)=\frac{j}{2}\left[\frac{z}{z+1+j}-\frac{z}{z+1-j}\right]$$

We have a set of complex conjugate poles at $z=-1\pm j$. Perhaps we will achieve more satisfying results if we work with this equation in polar form. We convert the two poles from Cartesian to polar form to get

$$z_1=(-1-j)=\sqrt{2}e^{-j\left(\frac{3\pi}{4}\right)}=re^{-j\theta} \qquad \text{for } \left\{r=\sqrt{2},\ \text{and}\ \theta=3\pi/4\right\}$$

$$z_2=(-1+j)=\sqrt{2}e^{+j\left(\frac{3\pi}{4}\right)}=re^{+j\theta}=z_1^*$$

Note that in polar form the poles are still conjugates of one another. If we rewrite Equation 4.57 in polar form, we arrive at

$$H(z)=\frac{j}{2}\left(\frac{z}{z-re^{-j\theta}}\right)-\frac{j}{2}\left(\frac{z}{z-re^{+j\theta}}\right)$$

Now we can use the same transform pair from Table 4.1 to get the inverse transform—only this time in polar notation. The new discrete time sequence in polar notation is given by

$$h(n)=\frac{j}{2}\left(re^{-j\theta}\right)^n-\frac{j}{2}\left(re^{+j\theta}\right)^n$$

Equation 4.60

We still have the nasty j in the equation, but this time we have an equation that looks a bit familiar and therefore it looks to have some promise in terms of simplification. Let's begin by rearranging and combining terms to get

$$h(n)=\frac{jr^n}{2}\left[\left(e^{-j\theta}\right)^n-\left(e^{+j\theta}\right)^n\right]$$

We can move the index n inside the parenthesis to get

$$h(n) = \frac{jr^n}{2}\left[e^{-jn\theta} - e^{+jn\theta}\right]$$

If we multiply both the numerator and denominator by $2j$ and bring the minus sign out of the brackets, we arrive at

$$h(n) = \frac{-jr^n}{2}\left[\frac{e^{+jn\theta} - e^{-jn\theta}}{2j}\right]2j$$

We can combine terms and use Euler's equation to arrive at

$$h(n) = -j^2 r^n\left[\frac{e^{+jn\theta} - e^{-jn\theta}}{2j}\right] \quad = \quad r^n\left[\frac{e^{+jn\theta} - e^{-jn\theta}}{2j}\right] \quad = \quad r^n \sin(n\theta)$$

If we replace the magnitude term r with $\sqrt{2}$ and the phase term θ with $3\pi/4$, we end up with an inverse transform that is a bit more pleasing to the eye. After all this fun work, we end up with the following:

$$h(n) = r^n \sin(n\theta) = \left(\sqrt{2}\right)^n \sin\left[(3\pi/4)n\right]$$

Equation 4.61

When we did the Cartesian to polar conversion, you probably noticed that the magnitude of the conjugate poles was greater than unity. Therefore the system described by the transfer function $H(z)$ should prove to be unstable because its poles are not contained within the unit circle. We can easily verify this by generating a bunch of values of $h(n)$ on an Excel spreadsheet. In this example, plots were made for 128 values of the index n. In order to increase the plot resolution and smooth the resulting curves, the index was incremented in steps of 0.1. Figure 4.18 shows the plot over 128 samples for the $\sin\left[(3\pi/4)n\right]$ term in $h(n)$. Figure 4.19 shows the plot over the same 128 samples of the magnitude or r^n term in $h(n)$. The composite plot over the 128 samples of the complete discrete time domain sequence $h(n) = \left(\sqrt{2}\right)^n \sin\left[(3\pi/4)n\right]$ is illustrated in Figure 4.20.

Finally, just for the heck of it, Figure 4.21 shows the two terms plotted separately but superimposed on one another. The trend illustrated by the plots is clear. The discrete time sequence grows without bounds. It fails the convergence test, and the system is unstable.

The reader should compare these plots with the stability plots shown in Figure 4.9 and Figure 4.10. The complex pole in this example was on the left-hand side of the z-plane, and it was outside the unit circle. Therefore we should have envisioned a discrete sinusoidal time function with an increasing amplitude. When you are deriving the inverse z-transform, a comparison of the function poles with those shown in Figure 4.9 and Figure 4.10 should provide you will all the information you need to predict the behavior of the final discrete time domain sequence.

This is all very interesting stuff. So why stop here? Let's do one more example. In the third and last example, we will take a close look at how to handle second order poles in a partial fraction expansion.

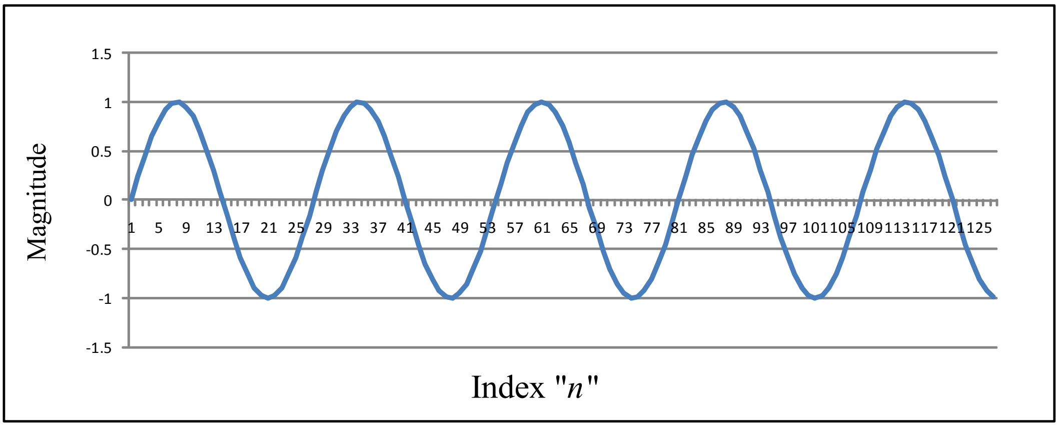

Figure 4.18 Plot of the sine term for $h(n)$

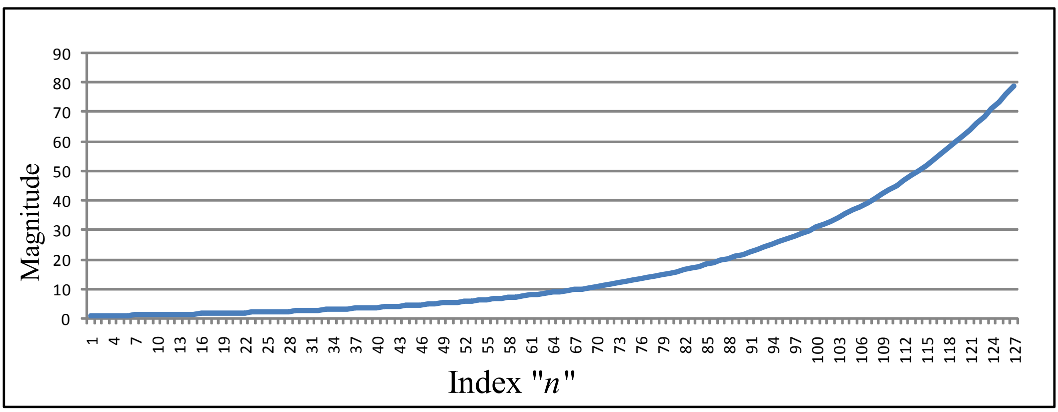

Figure 4.19 Plot of the magnitude term for $h(n)$